Numerical Study of the Motion of a Simple Nonlinear Pendulum

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Introduction

The purpose of this project was to create a program to find numerical solution for the equation of motion of simple nonlinear pendulum. To do this the RK4 numerical method was used which was created by C. Runge and M. W. Kutta to solve nonlinear differential equations. With this method the position of the pendulum and velocity of the pendulum as function of time was found. Once having this information the period of the pendulum from the numerical values can be calculated and compared to analytical values from the simple harmonic motion equation.

Method

As a starting point CC-programming language code for RK4 method subroutine has been adopted from Ref 3. The code was modified and used in the program to solve equation of motion for a specific size pendulum. It has been accomplished by calculating the moment of inertia for the pendulum and using that value to find the restoring force coefficient in the equation of pendulum’s motion. The program has allowed for changes in initial setting for the position and angular velocity. By changing these initial conditions it has become possible to see how the pendulum’s period was affected, especially when the amplitude of oscillations has become large enough to observe nonlinear effects.

Using three different analytical approximations for solutions of nonlinear equations of motion, we have compared these approximations to the numerical results to see how accurate the analytical approximations were.

The first analytical method we have used was the simple harmonic motion without accounting for the nonlinear effects.

The second analytical approximation was expansion of elliptical integrals which takes into account the increase in period as the initial angle increases. We used the equation $T = T_0(1 + \frac{1}{2!} q^2 + \frac{3}{4!} q^4 + \frac{5}{6!} q^6)$, where $\theta$ is the initial angle and $T_0$ is the period of simple harmonic motion.

The third method, known as the Ritz method, is based on minimization of the action integral of the pendulum. Action is the integral of the Lagrangian with respect to time. Variational procedure has been used to minimize action integral with respect to the amplitude and then solving for the angular frequency of the pendulum. The angular frequency came out to be $\omega^2 = k(1 - \frac{L}{2} cos^2 \theta)$ where $k$ and $L$ are parameters dependent on the pendulum’s shape and $\omega$ is the amplitude.

Finally we have compared how the phase diagram changed as the initial angular velocity increased. Then by modifying the equation of motion to include a drag force which equals $F = -bv$ and see what happens to the phase diagram.

Results

From the data collected a graph has been made with the different calculated values of the pendulum’s period for all the methods described. The simple harmonic motion method didn’t account for the increase in period as the initial angle increased. The expansion of elliptical integrals was the most accurate method used, where the values calculated were very close to the numerical values. The Ritz method started off very accurate until the initial angle became too large (see the graph below).

Then the phase diagrams for the different values of initial angular velocity were graphed. As the angular velocity reaches its highest value the graph becomes less elliptical shaped. (see the graph below)

Then a drag force was added to the program to find our final phase diagram. (see the graph below)

Discussion

Overall the expansion of elliptical integrals was found to be the most accurate analytical method for finding period.

The Ritz method coming in second because the first order approximation for simplification was used since the second order approximation is extremely difficult to calculate.

As for the phase diagram without drag force, the graph shapes became less elliptical due to the increase of the pendulum’s period. This means the larger the initial angular velocity the longer it takes for the momentum to change directions.

Finally the drag forces have been added to the program and the phase diagrams have been studied. It has been found that as the drag force decreases the angular velocity of the pendulum and the momentum of the pendulum starts to decrease as well.

References