The Pedagogy of Mathematical Logic

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I fell in love with education at a young age; I have my mother to thank for that. I would like to take a moment to thank her for her letting me play teacher in her classroom every day after school growing up, and instilling in me the belief that I could be anything that I wanted to be. What she didn’t know was that who I wanted to be stood right before me, I wanted to be just like her. Whenever I became a bit older, she even provided real papers for me to grade. I am not sure who really benefitted from this more. Thank you Mom, I love you always.
Abstract

In higher level mathematics there is a great shift from concrete computational mathematics to abstract theory mathematics, namely mathematical proofs and logic. A world that is exciting, frustrating, and rewarding. It can propel students forward into abstract, critical thinking that will transfer to multiple facet of their life. With such a vital foundation before us we are left with many questions, such as: how can this foundation be concreted in students minds? Can it be attained by traditional higher education methods? What would contemporary education methods (created by Harry Wong, Spencer Kagan, etc) have to say about teaching this topic? Just how far does this curriculum reach into other disciplines?

This thesis attempts to remedy some of these questions by an in depth look at the fundamentals of logic, where they come from, just how far they reach into other disciplines along with a research based exercises and educational strategies to tackle these concepts. These are done while considering educational psychology, current curriculum and instructional shifts and trends in higher education, and the pressing curriculum at hand.
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Chapter 1:

Introduction

“Contrawise, continued Tweedledee, “If it was so, it might be; and if it were so, it would be, but
as it isn’t, it aint. That’s logic.”

From Lewis Carroll’s *Through the Looking Glass*

Logic is essential to higher mathematics. Through mathematical logic anything is possible. It is
an endless puzzle of possibilities. However it is a realm that few tap into and even fewer fully
embrace. This approach to mathematics scares many students because a computational answer
does not suffice. It is a place where philosophy and mathematics collide and mathematical
theorems are created. For example, any middle school geometry student will know the formula
\[ a^2 + b^2 = c^2, \] aka the Pythagorean Theorem. We trust the application of the Pythagorean
Theorem on every right triangle we encounter and we know that the Pythagorean Theorem works
on all the triangles created thus far, but we can’t possibly create every right triangle. We could
toil and toil, but not all the mathematicians in the world working in unison could reach the goal.
How do we know for certain that there is not a right triangle out there that does not follow the
Pythagorean Theorem? Is there a right triangle in existence that defies this, and if there is, what
does that say about any discoveries involving the Pythagorean Theorem? Are they also invalid?
What would this say about mathematics as a whole?
There must be some underlying structure, some laws that govern mathematics that make it true and trustworthy. This is precisely where mathematical logic and proof step in and save the day (no cape or white horse needed).

There are multiple different proofs of the Pythagorean Theorem, in fact, James Garfield (a former United States President) offered his take on the Pythagorean theorem through a trapezoidal representation. The oldest known proof is credited to the Greek Philosopher Pythagoras (590 B.C.E). It is summarized shortly below.

![Figure 1.1 Proof of the Pythagorean Theorem](image)

**Figure 1.1 Proof of the Pythagorean Theorem**

Proof:

Suppose you have four congruent right triangles with sides A, B, and hypotenuse C.

Let them be arranged end to end with a rotation of $90^\circ$.

This yields a square with sides = A+B and a smaller square within with sides, C. (See figure 1.1)
The area of this figure can be calculated two separate ways and that area must be equal to each other. 1) The first using method utilizes the formula for Area of a quadrilateral:

\[ \text{Area} = (\text{length})(\text{width}) \]

and 2) the second is calculating the sum of the small square within the figure and the four small triangles around the outside of the figure.

1) The formula for area of quadrilateral is: \( A = (\text{length})(\text{width}) \)

Therefore the area of this our figure is \( \text{Area} = (A+B)(A+B) \)

Using the distributive law of multiplication this becomes \( \text{Area} = A^2 + 2AB + B^2 \)

2) Calculating the sum of the small square within the figure and the four small triangles around the outside of the figure uses the same formula for a quadrilateral, \( A = lw \) except in this scenario length and width are equal to the hypotenuse of our triangles, \( C \).

So, Area of the small square = \( C^2 \)

The formula for area of a triangle is: \( \text{Area} = \frac{1}{2} (\text{base})(\text{height}) \)

As stated whenever first creating the figure the triangle has sides, \( A \) and \( B \).

Therefore \( \text{Area of the four triangles} = 4( \frac{1}{2} AB) = 2AB \)

Adding these two areas together yields: \( \text{Area} = C^2 + 2AB \)

The two calculated areas must be equal since the same figure is used so,

\[ A^2 + 2AB + B^2 = C^2 + 2AB \]

By the addition property of equality we can obtain the equation
This equation is also known as the Pythagorean Theorem.

Q.E.D.

Through logic we can calculate the hypotenuse of any right triangle when given two sides with confidence! More importantly a whole world of mathematical theorems and proofs are available to anyone willing to navigate through it!

Another problem now arises. How do we as educators excite students about this great shift from computational to mathematics logic? How can we challenge students to step outside of their comfort zone of a solid computational answer and into a land of deep thinking and theories. I believe it is possible. We must excite students about the journey from a simple mathematical thought to a solid mathematical proof and the potential that they themselves hold. I believe we must give them the tools to construct a solid mathematical proof and let them build the house.

“Every mathematical problem is solvable…We always hear the cry within us: There is the problem, find the answer; you can find it by just thinking, for there is no ignorabimus in mathematics” –Hilbert (1925)

This paper will not only address the complexity of mathematics logic, but present some educational assignments to aid students in discovering mathematical logic. This It will be broken into 3 sections: 1) A Brief Introduction to the World of Mathematical Logic 2) Higher Education and The Goals of The Pedagogical Pathways 3) Pedagogical Pathways.

_A Brief Introduction to the World of Mathematical Logic:_ In this section I will briefly discuss the history of Discrete Mathematics. How can we know where to go if we do not know where we
have already been? I will also touch on the applicability of mathematical logic and a few avenues where a solid logic foundation is vital outside of the world of mathematics.

*Higher Education and the Goals of the Pedagogical Pathways:* In this section I will present 14 different questions I posed in creating the pedagogical pathways presented in the third section. These questions were formulated from a compiling of educational research and will guide the pedagogical pathways section of this paper.

*Pedagogical Pathways:* In this section I will present different methods of tackling the basic concepts of discrete mathematics. These assignments introduce and provide supplementary activities to aid in the teaching of the basics of mathematical logic. They address the following topics: It’s all Greek to Me: *Symbolism and Foundations of Mathematics Logic*, The Truth about Truth Tables, The Road Before us and the Road Behind: *History and Research Applications*.

So come away on this journey with me, I hope that it leaves you inspired and excited about the possibilities of your students and ignites the flame of future exploration. In the least I hope it is a pleasurable read and leaves you with a richer knowledge of discrete mathematics in the classroom.
Chapter 2:

A Brief Introduction to the World of Mathematics Logic

The history of logic can be traced back to ancient civilizations, since the beginning man has been interested in the natural world and why things are the way they are. We are naturally curious beings. Society and nature operate on a mathematical system therefore mathematical system also has a deep history. The intersection of logic and mathematics however has a slower moving and more recent history. In the sixth century B.C. the Greeks began to insist that geometric statements be established by careful deductive reasoning rather than by trial and error (Greenberg, 2008). This was necessary because there were many mathematical theories at the time. Some of these theories are still accepted today but others seem completely off base. “For example, according to the Rhind papyrus, written before 1700 B.C. by the Egyptian priest Ahmes, they thought that the area of a circular disk was equal to the area of a square on the eight-ninths of the diameter” (Greenberg 2008). This was an accepted method even though to the modern day mathematician it seems preposterous.

There needed to be a structure an underlying system, something to take this idea of mathematics and structure it into a logical organized system accepted by all. No physical experiments could be performed to verify that the statements about ideal objects are correct – only the reasoning in the demonstrations can be checked. This is the root from which Euclidean geometry grew. Also, at this time in Greek history “among Greek philosophers, dialectics, the art of arguing well, played an important role” (Greenberg, 2008). This sets up our two schools of thought which remained separate until the 16th century.
Axiomatic geometry and philosophy were taught separately in the schools of the era, according to the Boethian division of knowledge, logic in the ‘trivium’ and mathematics in the ‘quadrivium’ (Magnia, 2010). These two schools of thought were in essence Aristotelian philosophy and Euclidean mathematics, they were peanut butter and jelly, and in the 16th century we discovered just how much they coincide.

“A philosopher who has nothing to do with geometry is only half a philosopher, and a mathematician with no element of philosophy is only half a mathematician. These disciplines have estranged themselves from one another to the detriment of both.” Frege

We will discuss the two schools of thought separately and then sticking with the sandwich analogy; explore the bread that brings them both together.

2.1 Euclidean Mathematics

Euclid is a familiar name to most anyone who has taken a high school geometry course. His influence stretches back to our earliest understanding of shapes in grade school. “He lived taught and founded a school in Alexandria on the western edge of the Nile Delta” (Derbyshire, 2006). Even though he lived in modern day Egypt he is thought to have gotten his mathematical training from Plato’s Academy in Athens Greece. In his most famous work, The Elements, written around 300 B.C (Greenberg, 2008) he set the foundation for the Euclidean geometry still taught today alongside non-Euclidean geometry. The part of his work that is most beneficial to this paper is the axiomatic way that Euclid unpacked geometry. “The axiomatic method is a method of proving that the results are correct and organizing them into a logical structure”
Euclid built on the achievements of his predecessors; the Pythagoreans, Hippocrates, Archytas, Eudoxus, and Theaetetus.

“Euclid’s Elements is not just about geometry and number theory; it is about how to think logically, how to build and organize a complicated theory, step by logical step.” Marvin Jay Greensburg

He started with the five most basic axioms/postulates and built geometry on them, I have paraphrased them below with a diagram for readability:

**Euclid’s Postulate 1:** For every two non-equal points, $A$ and $B$, there exists a unique line passing through $A$ and $B$.

![Figure 1: Euclid’s First Postulate](image1.png)

**Euclid’s Postulate 2:** For every two line segments, $AB$ and $CD$ the line segment $AB$ can be extended to a point $E$, such that $B$ is between $AE$ such that $BE$ is congruent to $CD$.

![Figure 2: Euclid’s Second Postulate](image2.png)
Euclid’s Postulate 3: For every two non-equal points, $A$ and $B$, there exists a circle with $B$ as its center and radius $AB$.

Figure 3: Euclid’s Third Postulate

Euclid’s Postulate 4: All right angles are congruent to one another.

$\angle ACB = \angle DCA$

Figure 4: Euclid’s Fourth Postulate

Euclid’s Postulate 5: For every line, $A$, and every point, $B$, not lying $A$, there exists a line, $C$, passing through $B$ that never intersects with line $A$. (In other words, $C$ is parallel to $A$).

Figure 5: Euclid’s Fifth Postulate: The Parallel Postulate
From these four postulates Euclid built the entirety of his geometry. He worked through the mechanics of geometry, always returning back to his postulates to rely on. Though there is some controversy over fifth postulate, especially with the rise of hyperbolic and elliptical geometries, Euclid created the foundation of geometry that student’s learn today. He practiced a method helpful to our students called ‘Lullism’ this term comes from the medieval philosopher Robert Lull (1232-1316) (Magnia, 2010). He believed that each complex concept that we conceive is analyzable into all its component parts, down to the simplest ones.

“One reason the Elements is such a beautiful work is that so much has been deduced from so little!” Marvin Jay Greensburg

It is important to note that during the Renaissance scholasticism was fiercely rejected and in particular against Aristotelian logic. As a result the standard of rigorous reasoning began to be identified with Euclid’s Elements.

“The Dialectician call ‘proof’ a syllogism which causes knowledge, i.e. one that concludes from proven premises; and this originates from Geometry. Better: every proof which leads us to truth is geometrical in character. As is very truly said, we are not able to distinguish the true from the false if we have not previously been well acquainted with Euclid.” Jacob Pelletier (1515-1582) Introducing his Latin translation of the first 6 books of The Elements (1557).
3.2 Aristotelian Logic

In the other corner we have Aristotle. Aristotle is credited with being the first to tack the formal study of logic. “Aristotle’s logic and metaphysical works contain elements of three distinct types of formal theory: ontology, a theory of consequence, and a theory of reasoning” (Thom, 2010).

Aristotle’s ontology worked in the metaphysical, the branch of philosophy that deals with being, knowing, and cause (Dictionary.com). He kept his ontology quarantined from his theory of logic. This allowed him to make many advances without contradiction, other philosophers of his time encountered trouble in this respect.

“The theory of consequence is about logical relations that connects propositions to one another” (Thom. 2010). He worked with syllogisms and the idea that there were four fundamental types of propositions.

- Universal, affirmative ("All X is Y")
- Particular, affirmative ("Some X is Y")
- Universal, negative ("No X is Y")
- Particular, negative ("Some X is not Y")

(Derbyshire, 2006)

Aristotle and logicians at the time generally used these in sets of three in three. They would combine statements to form logical relationships, for example;

- All X is Y
- All Y is Z
- So All X is Z
Syllogisms occur where a conclusion is drawn from two given or assumed propositions (premises). This deductive reasoning is commonly used in mathematical proof. Aristotle worked with multiple syllogisms and the different properties of them.

Many of the properties of Syllogisms Aristotle is credited with can be directly related to a mathematical proof. One of these properties common to mathematical proof is the properties of deducing a conclusion from a false premise. Given from a syllogistic perspective, Aristotle stated that “though syllogism is a necessary consequence, there are non-syllogistic necessary consequences” these non-syllogistic consequences can occur where the premises include extra material or leave out necessary information. Until these non-syllogistic additions and consequences are formulated into syllogistic consequences they are considered defective syllogisms. As earlier stated this property of syllogisms directly correlates to a property within a mathematical proof.

Truth tables are used commonly in mathematical proof to show validity of logic. A truth table assigns truth values of 0 (zero being false) or 1 (one being true) to different operations on given variables. Generally it begins with at least 2 premises (in our case, P and Q) and analyzes every possible truth value scenario (see columns 1 and 2 in figure 2.1). After the structure of the truth table is constructed, operations involving P and Q exclusively can be analyzed.

As stated, in a mathematical proof, deducing a conclusion from a false premise is invalid. For example:

Let’s use the premise, 2+5=3.

This premise is obviously false, but we assume it is true.
Then, by using the addition property of equality, adding one to each side, we can deduct that \(2+6=4\).

This conclusion is also obviously false, but by way of logic and proof no laws were violated. The only misstep was allowing our premise to be false.

The logic used above symbolically is \(P \rightarrow R\) (P implies R), we have a premise and we are showing that R is a consequence of our premise, P is implying R. This property is very obvious in a truth table, it would be constructed as follows:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>(P \rightarrow R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Figure 2.1 Truth Table**

As you can see, when our premise, \(P\), and the conclusion, \(R\), have a truth value of zero, meaning that both statements are false, within the operation \(P \rightarrow R\) they lead to a truth value of 1. As seen above, this can be misleading in terms of proof. This is also true whenever extra material is presented within the premise, in Aristotelian terms, “Everything good is just, every virtue is good and the Sun is in Cancer, therefore every virtue is just” From Abelards, *Dialetica* (Thom, 2010). The Sun is Cancer has nothing to do with the conclusion “Every virtue is good” so its appearance in the proof is arbitrary and could be misleading. He explores other types of syllogistic errors in his book *Prior Analytics*, many of these can be directly paralleled to mathematical proof.
This particular approach to Aristotelian logic is an example of demonstrative logic. “According to Aristotle, a demonstration is an extended argumentation that begins with premises known to be truths and involves a chain of reasoning by deductively evident steps that its conclusion is a consequence of its premises” (Corcoran, 2009). This should sound familiar to anyone who has done a mathematical proof. “He presented a general truth and consequence concept of demonstration meant to apply to all demonstrations” (Corcoran, 2009). Demonstrations are meant to produce knowledge just as a mathematical proof is meant to produce mathematical knowledge.

“Suppose I draw a line, how many points are on it? You’re likely to say infinitely many points. But Aristotle would say, none. However Aristotle maintained, if I make a cut mark in the line, I create a point. Now it has one point. I can do this over and over again, creating many points on the line and second, no matter how many points I have created, I can always create more. Thus, says Aristotle, the points on the line are a potential infinity, not an actual infinity.”

Haskell B Curry *Foundations of Mathematical Logic* (1963)

This is an example of an Aristotelian demonstration of a line. It is constructivist and very similar to the Euclidean approach shown above.

Another common method in mathematical logic today is a proof by contradiction. In Aristotelian logic this is called *indirect demonstration*. It is assuming first that the negation of a statement is true and showing that this leads to a contradiction of known premises. It operates under the assumption that if a premise is true, it’s negation must be false and vice versa. It is also known as a proof by *reductio ad absurdum*, abbreviated RAA. Therefore if we can show that the
negation of a statement is false then the original statement must be true. Hilbert and Ackerman later expanded on this indirect demonstration in 1928:

“The fact is, however, that the occurrence of a formal contradiction, i.e. the provability of two formulas $A$ and $\neg A$ would condemn the entire calculus as meaningless; for we have observed that if two sentences of the form $A$ and $\neg A$ were provable the same would be true of any other sentences whatsoever.”

Hilbert and Ackerman (1928)

Section 2.3 The Merging

So as you can see, our two schools of thought may not be in separate corners, they are on the same team, even though this teamwork was not yet developed during this time. It is true that Aristotle was not a mathematician but he was a philosopher and the converse true for Euclid. They both are said to have studied or had connections with Plato’s Academy and “Plato’s Academy, whose entrance is said to have carried the motto: “Let no one unversed in Geometry enter” (Corcoran, 2010). So Aristotle’s background included knowledge of geometric reasoning and vice versa. In the Republic, Plato wrote: “The study of mathematics develops and sets into operation a mental organism more valuable than a thousand eyes. Because through it alone can truth be comprehended” (Greenberg, 2008).

Still up until the middle ages “logic was considered a science of language” (Magnia, 2010). Consider the following paradox:
The council of a certain village is said to have given orders that the village barber (supposedly unique) was to shave all the men in the village who did not shave themselves, and only those men. Who shaved the barber?

From Foundations of Mathematical Logic Haskell B. Curry (1963)

As you can see these paradoxes are manipulations of words and language rather than mathematical objects. They are considered semantic paradoxes; the logic questions proposed at this time were of moral persuasion. What is good? What is evil? Whenever we speak of logic of language this is what they are referring to. These questions, though starkly different at first glance still use much of the same reasoning as mathematical and logical systems. “In the study of philosophical logic it has been found fruitful to use mathematical models” (Curry, 1963).

It wasn’t until the second half of the 15th century that the relationship between Euclid and Aristotle was explored in depth. Dasypodius and Herlinus ventured to make explicit the logical structures of each proof contained in The Elements. They worked through The Elements and constructed proof on the Aristotelian syllogistic standards.

Father Honore Fabry (1607-1688) is credited with some of the major advances in the way of proof consistency. He attempted to separate logic into artificial: dealing with determining the rules of valid inferences, and natural: dealing with the metaphysical. It is important to note that both the natural and artificial logic root in Aristotelian thinking. He concluded that logic has to take the structure of the axiomatic discipline like geometry to guarantee the maximum of rigor and certainty of the proofs (Magnai, 2010). Saccherri also believed this. In his work Logical Demonstritiva he writes “That rigorous method of proof which hardly allows first principles and
which does not admit anything not clear, not evident, anything which could be called into
doubt.”

As you can see from the middle ages onward there was a growing reverence for
Euclidean type logic for certainty in all areas. The cry for rigorous proof was heard throughout
the disciplines and philosophers and mathematicians alike recognize that “it would be a mistake
to suppose that philosophical and mathematical logic are completely separate objects. Actually,
there is a unity between them…Any sharp lines between the two aspects would be arbitrary”
(Curry, 1963).

2.4 The Formation of the Fundamental Proof Components and Symbolism

Now that the two disciplines commonality is obvious, the focus can shift from the
merging of classic logic and Euclidean geometry, to the emerging symbolism and proof structure
used today as a result of this merger. With the reverence of ‘Lullism’ began a use of algebraic
symbolism in proof. Francois Viete (1540-1603) was among them, he performed calculations
using letters of the alphabet to obtain a high level of generality. He aimed to use letters perceived
as assimilating algebra to an artificial language, a language particularly fit to express rigorous
logical arguments (Magnai, 2010). Trying to assimilate language and algebra can be very tricky.
Consider this paradox known as the Grelling Paradox:

Among English adjectives there are some, such as ‘short’ and ‘polysyllabic’ and
‘English’ who apply to themselves. Let us call such adjectives autological; all
others heterological. Thus ‘long’, ‘monosyllabic’, and ‘green’ are heterological.
Then if ‘heterological’ is heterological, it is autological, and vice versa.

From Foundations of Mathematical Logic Haskell B. Curry (1963)
The paradox above comes in attempting to label the group of heterological words. By labeling the group of all non-self-identifying words, the label, heterological, is now describing itself, making it autological. The word ‘English’ is written in English, therefore it is describing itself. English is an English word. The work ‘short’ is a small one syllable word, ‘short’ is a short word. The same can be said of ‘polysyllabic’, polysyllabic contains multiple syllables, making polysyllabic a polysyllabic word. The same is not true of the words ‘long’, ‘monosyllabic’ and ‘green’. The word ‘long’ is long syllable and contains four letters; ‘long’ is a short word. The word ‘monosyllabic’ contains multiple syllables and the word ‘green’ is in fact not a green word.

So, assume that heterological is a heterological word, it is therefore applying to itself. Heterological would then by definition be autological because it is describing itself. If we assume that the converse is true, that heterological is an autological word then by definition of autological it would be a word that describes itself. If heterological is describing itself, just as ‘short’ was a short word, heterological would be a heterological word, making it autological. So what is the word heterological? It cannot be both, it cannot be neither, and it cannot be just heterological or just autological. This is our paradox.

Below I have shown this same paradox using mathematical language, reasoning and symbols to make obvious the parallels between logic and mathematics. However, converting this to an algebraic language can get very tricky, it can be done but not without the use of quantifiers and symbolism. So for the ease of reading the proof, I have included the following table of symbols. See Table 1.
<table>
<thead>
<tr>
<th>Heterological</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>All non-self-identifying worlds</td>
<td>$\mathbb{H}$</td>
</tr>
<tr>
<td>All self-identifying worlds</td>
<td>$\mathbb{A}$</td>
</tr>
<tr>
<td>short, polysyllabic, English</td>
<td>$w, y, z$</td>
</tr>
<tr>
<td>long, monosyllabic, green</td>
<td>$s, t, u$</td>
</tr>
</tbody>
</table>

**Table 1. Language to Symbols Conversion**

So our original paradox can be written as follows:

Suppose that $\forall x \in \mathbb{A} \lor \mathbb{H}$

$\mathbb{A}$ being all self — identifying words and $\mathbb{H}$ being all non — self — identifying words

Let $w, y, z$ be the adjectives: short, polysyllabic, English

Observe that: $w, y, z \in \mathbb{A}$

Let $s, t, u$ be the adjectives: long, monosyllabic, green

Observe that: $s, t, u \in \mathbb{H}$

Let $q$ be the adjective heterological.

If $q \in \mathbb{H}$

By definition of $\mathbb{A}$

$q \in \mathbb{A}$

$q \in \mathbb{H} \land \mathbb{A}$

This violates our first premise: $\forall x \in \mathbb{A} \lor \mathbb{H}$

Let $q \in \mathbb{A}$
By the definition of $A$

$$q \in H$$

*This violates our first premise: $\forall x \in A \lor H$*

So $q \notin A \lor H$

*but $\forall x \in A \lor H$*

This is a contradiction

Through the use of mathematical proof, quantifiers, and symbolism the same paradox can be obtained. However, the symbols used above had to be created and agreed upon. Leibniz along with other mathematicians at that time, including Jacob Bernoulli, wanted to develop a symbolic language for reasoning with a simple set of basic rules and remove the ambiguity of natural language.

The art of combinations [. . .] signifies purely the science of forms or formulas, or even of variations in general. In a word, it is the universal specious arithmetic or characteristic. [. . .] One can even say that calculation with letters, or more precisely algebra, is in a certain sense subordinate to it, because one employs many signs which are indifferent [. . .] For this aim the letters of the alphabet are highly suitable. And when these letters or signs signify magnitudes or numbers in general, the result is algebra or rather Viete’s specious arithmetic. [. . .] Then, if these letters were to signify points (as this is effectively practiced by geometricians) one would be able to create a certain calculus or sort of operation which would be quite different from algebra. [. . .] When these letters signify terms or concepts, as in Aristotle, this gives us that part of logic which deals with
figures and modes. [...] Finally, when the letters or other characters signify true letters of the alphabet, or of the language, then the art of combinations together with consideration of the languages gives us cryptography.

Leibnez: *The Dissertation on the Combinatorial Art* (1666)

The notation mathematicians most commonly use today was not completed until the 1880’s with Guiseppe Peano and Mario Pieri. Augustus De Morgan also contributed to these efforts in 1833 with his work *Formal Logic, or the Calculus of Inference*. Most of this work was focused on *quantification of the predicate*, and improving the traditional way of writing out logical formulas (Derbyshire, 2006).

George Boole was also on the scene at this time. His most notable work was the ‘algebraization of logic’, where he is said to be “the man who married algebra to logic”. He wanted to convert logical statements to an algebraic language. His greatest supporters claimed that Boole had uncovered the essence of pure mathematics. His critics at the time believed that pure math is logic, so they saw nothing innovative in his work. However, what Boole had discovered was not pure mathematics; it was a new branch of applied mathematics, the application of algebra to logic. His method is still taught today, commonly in respect to computer science and their algorithms.

2.5 Modern Mathematical Proofs and their Applications

Since this time proof in mathematics has become a necessity for a mathematical theorem to be considered valid. Consider Goldbach’s Conjecture: it states that the sum of two prime numbers is an even number. This statement has been checked into the billions without a counter example but without a proof it is not considered a theorem or used to establish other theorems (Brown, 1999). “In a given mathematical system, the only statements we call *theorems* are those
statements for which a *correct proof* has been supplied” (Greenberg, 2008). Diagrams and pictures do not replace the role of a written proof because they can be misleading, even though proofs without words are of much interest in current mathematics and can offer insight into a mathematical system. Proofs must be based in classical two-valued logic; no proposition is both true and false. If a proposition is false then its negation is true. G.H. Hardy once said: “Mathematical Theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In some sense mathematical truth is part of our objective reality.”

In fact, mathematical proof has emerged as its own branch of mathematics. To name a few widely known areas of study: George Cantor and his development of set theory, advanced analysis, abstract algebra, there is even a branch of mathematics that creates and proves theorems about the entire branches of mathematics, it is called *metamathematical*. The possibilities and discoveries of mathematical logic truly are endless.

Mathematical logic is central to computer science. The values of true and false for statements correspond to the high and low voltages in the electronic circuits of a computer (Stone, 1973). Think from a programmers perspective for a moment. How is it possible to program a computer to play chess? A programmer must unpack every possible scenario within a chess board and provide a plan of action. For example: if a queen is exposed horizontally from a rook then move the rook and take the queen out of the game. It is an “if-then” statement just as in mathematical logic. Computer programmers must use logical systems to solve computer algorithms daily. In fact, truth tables and mathematical logic is introduced in the foundation courses of computer science and operation courses because it is vital to its study. Students will explore a bit about computer science and algorithms in the pedagogical pathways.
Knowledge of the history of mathematical logic, and analyzing it through the mind frame and thought processes of early mathematicians, can aid students in the early formation of mathematical thinking. It is also important to expose students to this vast history to help in comprehension and perspective of mathematical logic. ‘Lullism’ and induction are ways of thinking that expands into all sciences. “We tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true. But the inferring of premises from consequences is the essence of induction; thus the method of investigating the principles of mathematics is really an inductive method and is substantially the same as the method of discovering general laws in any other science” (Russell 1907). Math and its logical processes is the foundation for many other sciences and schools of thoughts. The same way a biologist observes the migration patterns of birds and the possible causes and mathematician can inductively proof a summation. “Mathematics hooks onto the world by providing representation in the form of structurally similar models” (Brown, 1999). Students will see elements of the philosophy of mathematics and the history of mathematical logic echoing throughout their mathematical career.
Chapter 3:

Higher Education and the Goals of the Pedagogical Pathways

3.1 Common Threads

3.1.1 Introduction

“Hey Mr. McCourt did you ever do real work, not teaching?”

“Are you joking? What do you call teaching? Look around the room and ask yourself if you’d like to get up here and face yourself every day.”

Frank McCourt *Teacher Man*

Teaching is no easy profession, but it is also one of life’s greatest privileges. Teaching is an opportunity to inspire students and help them unlock the potential that is all too often overlooked. The trick is, especially in higher education, making students the master of their own education. “Learners are not simply passive recipients of information. They actively construct their own understanding” (Svinicki, 1991). This is very true in the realm of mathematical logic because there is so much room for creativity and deep thought. There is no clear pathway, the students must work through each problem and think. “People are more likely to enjoy their education if they believe they are in charge of their own decision to learn” (Bain, 2004).

Mathematical logic gives students this opportunity. They can leave the classroom perplexed about a question, thinking through a new process making it their own, and most importantly responsible for their own learning. “Of the best leaders, at the end of the day the people say, we did it ourselves” Lao Tsu. The goal of the pedagogical pathways section of this
paper is discovery and learner oriented; students will hopefully feel like they discovered logic, that it is only a mere matter of years that is preventing them from beating Aristotle and Euclid to the punch.

In educational research today there are some common threads in regards to what the best teachers do which I will explore in depth below, these include: Relevant Instructional Time, Learner Centered Instruction, Creating a Safe Critical Learning Environment, Encouraging Collaboration, and Presenting Differentiated Instruction to Accommodate Multiple Learning Styles.

3.1.2 Relevant Instructional Time

Students must come to class and receive an education that is relevant to that meets them where they are at and applies to life as they know it and we know it. It must be applicable and relevant. Students will be less interested in a curriculum they feel they will never use again. R. Barker Bausell offers the following formula:

\[
Relevancy of Instruction \rightarrow More Time on Task \rightarrow More Learning
\]

He believes that the relevancy of instruction, time on task, and learning are directly proportional. This includes tailoring curriculum to fit students, making it relevant to the classroom. Each student is different, there is no one size fits all education. If the students believe the curriculum is relevant they will be more willing to work and more learning will take place. Well organized material and clearly defined instruction are also crucial (Berrett, 2012). With something as foreign as mathematical logic it is imperative that students know what is expected of them and the material is well reviewed and organized. This will increase time on task and help keep the students focused.
3.1.3 Learner Centered Instruction

Behaviorist Ivan Pavlov is famous for his Theory of Classical Conditioning. In a nutshell this experiment used a dog, a bell, and food. Every day he first rang the bell and then presented the dog with food. The dog then started associating the bell with food and eventually Pavlov was able to ring the bell and the dog would begin to salivate even if no food was offered (Slavin, 2009). This is an example of one of the great pitfalls in education today commonly referred to as the “plug and chug”. Students after hearing a lecture and seeing examples will eventually be able to replicate the teachers result on the board correctly; teachers can condition the students to do a problem by using formulas and repetition “but is this really learning? I would challenge this statement by saying no. I would consider this to be survival” (Jackson, 2012). The “plug and chug” is especially easy to fall into in mathematics because, in most cases students are presented with a formula, they plug in given numbers, and away they go; eventually arriving at a computational answer that is either right or wrong. They may not know how or why they stumbled upon this answer, but they followed the steps that were given to them. If they were given these steps as part of a multi-step critical thinking problem where the steps are not easily recognizable could they solve it? If a student does not know the ‘why’ how can they know the ‘when’ to use it?

What is richer, knowing the slope of a line given two points using the formula \( m = \frac{y_2 - y_1}{x_2 - x_1} \) or knowing that the slope of a line can be determined from a formula but it also correlates directly to the graph, it is the m in the formula \( y = mx + b \), is also known as the rate of change, is also a ratio, and knowing how to manipulate it (i.e. make it steeper or flatter, etc.). It is easy to
see that there is much more to the slope of a line than a simple formula, but many students miss this correlation because they are too busy “plugging and chugging.”

In a learner centered classroom everything is done to benefit the learner and help them realize the ‘how’ and ‘why’ rather than the ‘what’. “In a curriculum designed to foster understanding, our students really do need to know what these things mean, where they come from, and how they fit into the grand scheme of things we call mathematics, one of mankind’s greatest intellectual achievements” (Marshall, 2012). Knowledge is constructed not received (Bain, 2004). In a classroom where students are free to discover a nugget of knowledge and carve their own name into it, make it their own it truly becomes a treasure.

3.1.4 Creating a Safe Critical Learning Environment

A safe critical learning environment is one where student’s feel safe not only in respect to their physical needs but also intellectually. They have the confidence to explore and create without the bonds of judgment or fear of failure. Creating this is especially vital in logic, because there is no formula for logic. Students will have to rely on their own understanding skills to fix a problem. The only thing educators can do is fill their tool box with all the necessary instruments. Students will be bringing incorrect mental models with them into the classroom, assignments that are crafted in a way that allows students to try their own thinking, come up short, receive feedback and try again can aid in slowly leading students to correct their own mistakes (Bain, 2004). Rome was not built in a day and mental models are not changed overnight. The students are being led on a journey; they will unlock the doors of their potential along the way. Mathematical proof is an art, an art of judgment and an art of discovery.
“The main focus of the course is on developing creative people, giving them confidence in themselves. We are not trying to force you into some kind of mold: on the contrary, we are trying to help you escape”

Paul Blake.

3.1.5 Encouraging Collaboration

“Tell me and I forget. Teach me and I remember. Involve me and I learn.”

Benjamin Franklin

Students feel most empowered and responsible in their learning whenever they have created it. Students bring all kinds of insight and good reasoning skills to the classroom with them and whenever they can be combined with other students’ strengths the results can be astonishing. “The atmosphere conducive to collaborative group learning is creating an environment in which individuals work together to construct new knowledge” (Cranton, 1996). Group work and collaboration when done well forces students into critical thinking but also gives them confidence that they are not alone, it is less intimidating than individual work.

More and more education research is pointing to collaboration as a key to a critical learning environment and learner centered classroom. There are a few guidelines that must be followed. Encouragers of a healthy collaboration include: flexibility, innovation, mutual respect and validation, acceptance, committing to common goals and a mutual vision (Schwartz & Triscari, 2011). These can be obtained by well-organized assignments and a natural critical learning environment where students feel safe to explore and a mutual respect of each other’s opinion is stressed. The most common inhibitors to healthy collaboration are distrust, competitions between
partners, education preparedness and partnering to camouflage deficiencies rather than capitalize on strengths (Schwartz & Triscari, 2011). It is important to acknowledge these in order to prevent them and/or recognize them if they are present to correct them promptly. “What begins to emerge is a model of education in which learners do more than accumulate information; they undergo deep-seated changes, transformations that affect both the habits of the heart and mind and the capacity for continued growth” (Bain, 2004).

### 3.1.6 Differentiated Instruction to Accommodate Multiple Learning Styles

“There is no one-size fits all approach to adult education” (Jackson, 2012). Students come from different backgrounds, creeds, and cultures; an educator who tries to reach all types of learners is going to be more successful than one who tries to force a square peg into a round hole. “Differentiated instruction not only accommodates all types of learning, but also aids in helping keep students engaged within their learning. “Really deep learning is a process that inevitably is driven by the learner, not by someone else. And it always involves moving back and forth between a domain of thinking and a domain of action. So have a student passively taking in information is hardly a very good model for learning. It’s just what we are used to” (O’Neil, 1995).
3.2 Preliminary Planning Questions

3.2.1 Introduction

This chapter is broken into 14 questions encountered whenever creating the pedagogical pathways. These are the major themes of the unit plan and should be evident throughout the activities. They are from a variety of sources, and they cover many different aspects of a unit plan and can be used whenever planning any unit. In short, the pedagogical pathways chapter/activities is approached “like a great meal, simply inviting students to the dinner table…what begins to emerge is a model of education in which learners do more than accumulate information; they undergo deep-seated changes, transformations that affect both the habits of the heart and mind and the capacity for continued growth” (Bain, 2004).

3.2.3 What big questions will this unit help my students answer, or what abilities or qualities will it help them develop {GOALS OF UNIT}?

This unit will introduce students to mathematical logic. It will tackle the questions: What is a proof? What are the attributes of a valid proof and what causes its validity? What Symbolism and mathematic representation of elements are used in proof writing? Are there laws governing proofs? It will not only solidify the basic concepts but also help them fully grasp reasoning and discoveries through critical thinking and the realization that mathematics in endless and exciting. They will be able to reason through a mathematical problem and construct a solid proof using symbolism, quantifiers and logic.

“Logical Necessity is not like rails that stretch to infinity and compel us always to go in one and only one way: but neither is it the case that we are not compelled at
all. Rather, there are the rails we have already traveled and we can extend them beyond the present point only by depending on those that already exist. In order for the rails to be navigable they must be extended in smooth and natural ways; how they are to be continued is to that extent determined by the route of those rails which are already there.”

Barry Stroud (1965)

3.2.3 How will I encourage my students’ interests in these questions and abilities?

What are some cross curricular opportunities that I can implement into this unit plan?

There are some obvious cross curricular activities in the world of mathematical logic. There is an evident parallel with computer science. Students working with truth tables and assigning the truth values 0 and 1 can switch over and work on a computer circuit. Also traditional logic can help formulate many algorithms in computer science. Historically, students will explore the origins of this thinking and step into the shoes of early mathematicians. This perspective will allow students to connect logic to something bigger than just mathematics.

3.2.4 What reasoning skills/background knowledge do my students need to have or develop in order to be successful in this unit?

Students need to have some concept of advanced mathematics in order to be successful. In the least a solid foundation in algebra is a necessity because it will be used in reasoning. More so, they need to have exposure to critical thinking problems and have experience reasoning through word problems and mathematical systems. A critical learning environment is key. Students need to know how to think on their own and not be frightened by a realm where a
formulas can’t solicit an answer, the answer is within them. “The method matters far less than do the challenge and permission for students to tackle authentic and intriguing questions and tasks, to make decisions, to defend their choices, to come up short, to receive feedback on their efforts and try again” (Bain, 2004).

3.2.5 Are there any incorrect mental models that students may bring with them to this unit that I want to challenge? How can I help them construct that intellectual challenge?

As stated earlier, students can be very alarmed by logic. It is not computational; there is not a definitive answer. Students can do a proof 4 different ways, each one just as correct as before. For students that are used to finding an answer the switch over to logic can be almost traumatic. Students should be eased into this switch, show that logic is not something they haven’t been dealing with their entire life. Non-mathematical elements will be used to introduce logical thinking such as a chessboard, a court briefing, a Sudoku puzzle, street signs and a checker board. The aim of this is to bridge the gap between the logic they already know and use they just do not realize it, to a solid mathematical foundation. The transition will seem natural and obvious to students.

3.2.6 What information will my students need to understand the important questions from this unit? How will they obtain that information? What questions/assignments can I present to aid in students in their understanding?

Students will need to understand the notation of logic to be able to fully analyze a mathematical proof. It is where symbols, sentences, and algebra converge. I propose in the Pedagogical Pathways section, that some of these new symbols, notation, and quantifiers be
taught using the same strategy that teachers use to teach reading in early childhood, just a more sophisticated approach. Assignments will be crafted to lead students into logical notation, and use that notation to construct a truth table. This will not only help them reason through a situation and certain outcomes from premises but also point on some ways that seemingly perfect logic can be flawed. They will confront flawed logic head on in an assignment called “A Case Brief.” In this assignment students will follow the standard format of a law student writing a case brief to analyze and proof and conclude whether or not the logic is sound within the proof. They will also have to research and work heavily in collaborative groups to feed off each other’s ideas and thoughts in mathematical logic. Most of the assignments in this unit are posed to help students discover mathematical logic and reason through for themselves why certain concepts make sense and fit where they do. For example: they will be constructing the constitution of mathematical laws from their own experience and truth table building instead of being given a list of laws. There are also ample opportunities for concept mapping that challenge students to bring all of the terms and concepts we are encountering, together into one web of interconnection.

3.2.7 What can I do to create a natural critical learning environment that is safe, fascinating and beneficial to students’ progress?

Mathematical logic is very abstract; therefore it can be a frightening step for students if they do not have a critical learning environment. Students must know that it is ok to be wrong, that there is no clear right or wrong answer. There are standards that guide proofs but two completely different structured proofs can solicit the same result. That is the beauty of mathematical truth, no matter where one travels; they will find their way to the same destination. Student’s need to
be asked tough questions, grapple with them, collaboratively and individually, receive feedback and guidance, and then start again.

Logic is not limited to a mathematical world, it is a logical progression, a way of thinking that students need to explore and work through in order to be better-rounded in all facets. “We need to design our teaching so that students can take the concepts of mathematics forward and apply them to each new situation they meet” (Marshall, 2012). In the pedagogical pathways of this section I am proposing a journal where students grapple with tough questions from each section, they can receive feedback immediately from the professor and work through them without feeling pressed or embarrassed. I am also proposing a lot of learner centered and collaborative learning assignments.

3.2.8 How will I confront my students with conflicting problems and encourage them to grapple with the issues? What questions in this section will students benefit from collaboration?

Throughout the pedagogical pathways students will be given collaborative projects. Students have to take logic and make it their own, a learner centered curriculum will encourage this. They will be creating the constitution of logic as a class, given questions such as:

- Why must there be laws of logic?
- Do we need proof in mathematics?
- Is there a parallel between mathematical logic and philosophical logic?
- What are the similarities? What are the differences?
- What is a number?
- What is an equation?
• What is a line?

• What compelled early mathematicians to try and tackle such a thing as

They will work as a team to solve logic puzzles, work through a new vocabulary and symbolism and make connections, explore concepts such as contrapositive, converse, inverse of, negation and logical equivalence and build background knowledge of these terms through discovery. They will walk a mile in the shoes of Aristotle, and also Euclid and his four postulates, constructing basic geometric proofs from his axiomatic system. It is important to note that most of the exercises within the pedagogical pathways could be tailored for individual or collaborative work depending on your class climate.

3.2.9 How will I find out what they know and what they expect to learn from this unit? How will I reconcile any differences between my experience and theirs?

If students are going to take responsibility for their own learning, it is important for the expectations of the course to be very clear. Also, if students feel like they have some control of the curriculum and course of study they will be more excited about the course. This class is like a great feast, I am inviting students to the table to dine, not forcing them to reluctantly eat their vegetables.

At the beginning of the unit I have included a short Pre-Unit quiz. It will reveal what preconceived ideas the students have about the course, how they think the world of mathematics and logic interact (if at all) and exactly what they expect from the course. This pre-unit quiz is less of “what the students don’t know but will learn” and more “what the students are already comfortable with, and expect from the course.” The students will also be given a list of areas to be covered in the course and be able to indicate what exactly intrigues them the most. One of the
beautiful things about discrete mathematics is its vast applications, if a particular topic covered intrigues a majority of students it may be beneficial to spin more lessons towards that application which is easily done.

Another policy that I am proposing is “Why Does It Matter?” (W.D.I.M.). This policy gives students the freedom to ask at any time “Why does this matter?” The curriculum must be relevant and the students must know why it is relevant in order for it to be effective. If a student feels like they have the freedom to question this it will not only help with making the curriculum relevant but also encourage a critical learning environment where students feel safe to explore.

3.2.11 How will I encourage metacognition and create better independent learners out of my students within this unit? What research based projects can I implement in order to propel students forward as individual learners?

Metacognition is formally defined as “knowledge about one’s own learning or about how to learn (McCormick, 2003). Essentially it is thinking about one’s own thinking. Students who practice metacognitive skills practice assessing their own learning, and taking steps to ensure maximum learning is occurring and remedying any information that is not clear using various learning strategies. These strategies include but are not limited to: Note-taking, underlining, summarizing, explaining understanding in writing, outlining and mapping, and the PQ4R method (preview, question, read, reflect, recite, and review). (Slavin, 2009).

These strategies are not only encouraged within the pedagogical pathways section but a majority of the assignments are directly designed to foster them. The notebook: “I’m 99% Sure This Notebook Exists” will not only act as a holder of notes and activities but also an interactive journal between the professor and student, a place where students are asked to
summarize data and explain their understanding of a topic in their own words. There are also fill in the blank notes that are will be stored in this notebook to help with note-taking strategies. Students will do concept mapping of key terms to help them realize the connections between all of the different aspects of discrete mathematics.

There is also a section of application research projects. This includes the final assessment, where students will analyze and write a case brief of the proof. In this project students will have to defend their logic in court case format. They will have to do some outside research and use many critical thinking in order to be successful in the project.

3.2.13 How will I communicate with the students in a way that will keep them thinking? What are some various ways that I can present information beyond oral lecture, including but not limited to kinesthetically, technologically, etc…?

In mathematical logic it is important to communicate with a student who may be operating within an incorrect mental model, in a way that fosters discovery learning. As I have previously stated there are many different roads to the correct answer in mathematical logic, this must be reflected in communication with students. Instead of telling students the exact way I would construct a proof, allow the students to explore different pathways and choose the one that fits their learning style best. If a proof was a concrete element sitting on the coffee table, the aim of this course is to pick that element up and examine it from all sides, gathering information with each turn.

There are many opportunities to get students up out of their seats and explore. First they will learn symbolism using some of the approaches used in teaching reading and phonics. The students must relearn that these symbols have meaning. They will not only be working
collaboratively, they will also be going out into the campus, finding street signs, and creating truth tables of the different scenarios presented by the sign. There are no two students who are exactly alike, so why do we as educators try to teach students of different backgrounds, cultures, creeds, maturity, and walks of life using one method? Any opportunity to get students engaged in their learning is an opportunity worth taking.

3.2.14 How will I spell out intellectual and professional standards I will be using to assess students work and why these are implemented?

Students need structure, it is traditional to place an assignment before students and require students to sponsor a definite answer. Once the assignment is received by the professor its’ answers are checked using a carefully and concisely constructed key to compare student answers with verbatim. I am proposing instead; giving students a rubric guide of standards you expect from them and letting them spread their creative wings and make a project their own. This is a performance assessment based approach to grading. The standards of each assignment in the pedagogical pathways are very clear. Students will know what is expected of them in respect to critical thinking and curriculum standards, but there is no one right answer to any assignment. This gives students a structure to operate within while exploring discrete mathematics. “A few well-thought-out, well-written items for a performance assessment could serve, for example, as a summative evaluation for all or most of your educational objectives” (Slavin, 2009).

3.2.15 How will I help my students who have trouble fully grasping these concepts?

One of the elements of creating a natural critical learning environment previously discussed was students receiving feedback and being allowed to try again. Students have the
opportunity to fail, dust themselves off and try again in many of the pedagogical pathways assignments.

3.2.16 How will I find out how my students are doing within this unit before formally assessing them? How will I give feedback separate pre-assessment? What are some various assessment strategies I will use within this unit plan?

One of the key features of the notebook is its potential as an assessment tool. This notebook is to be checked by the professor on multiple occasions in order to assess students understanding and comment on any questions the students may have left within their work/open ended questions. Professors will also be able to see where students stand on vocabulary and the how they map to a mathematical proof.

Within the pedagogical pathways the research based case brief uses all of the skills that would be used on a traditional formal assessment but in a more research based format. Many of the activities within this unit implement high level thinking that could be used as assessment level efforts.

3.2.17 What are my instructional goals of this unit? (Specific examples of: Upon completion students will…)

- Upon completion students will be able to recognize, interpret, and apply correct mathematical symbols and quantifiers.
- Upon completion students will be able to evaluate and construct a truth table.
- Upon completion students will be able to recognize and apply the Laws of Logic and Equivalence appropriately.
• Upon completion students will be able to construct a basic proof using Euclidean axioms.

• Upon completion students will be able to analyze a mathematical proof and apply the logic to verify that the proof is consistent.
Chapter 4: The Pedagogical Pathways

4.1 Introduction

Without further ado we have finally reached the Pedagogical Pathways. These assignments were created after the 14 questions in the previous chapter were addressed and considered. This chapter is broken into 3 subsections: 1) It’s all Greek to Me: Symbolism and the Foundations of Mathematical Logic 2) The Truth about Truth Tables 3) The Road Before us and the Road Behind: History and Research Applications. In each section you will find a vocabulary map, supporting assignments and corresponding Journals, Reflections and Ponderings. Each assignment is prefaced by an introduction of its own including: an abstract, student objective, materials needed, and an in depth overview. Each assignment also includes a blank assignment and a Sample Key. It is a sample key, because no two students will turn in the exact same assignments. Each assignment is open to creative interpretation and exploration by the students.

The notebook, I’m 99% Sure This Notebook Exists is a major component of the pedagogical pathways. This notebook is designed to enrich students learning experience in mathematical logic. The aim is to create a comprehensive portfolio of logic where students can organize their thoughts and ideals throughout the course. It is a sectioned notebook includes the following 3 sections: Vocabulary Mapping, Journals, Reflections and Ponderings, and the Constitution of Logic. These assignments are scattered throughout the pedagogical pathways and each one is labeled underneath the title with the words “To be included in notebook”. It is essentially a binder students keep throughout the semester with 3 major sections-tabs. For ease of organization, I have included an index of all assignments to be included in the notebook.
I’m 99% Sure This Notebook Exists

Index:

Tab 1: Vocabulary Mapping:

Vocabulary Maps are presented at the beginning of each section and cover major vocabulary and concept connections covered within that section.

- Vocabulary Mapping – Basic Terminology
- Vocabulary Mapping – Conjugate Propositions
- Vocabulary Mapping – Complete Concept Mapping

Tab 2: Journals, Reflections, and Ponderings:

Journals, Reflections, and Ponderings Assignments are presented at the end of each section and correspond to the assignments within that section.

- Journals, Reflections, and Ponderings Assignments

Tab 3: Constitution of Logic:

This is a major project presented in *The Truth about Truth Tables* Section of the Pedagogical Pathways.

- Laws of Logic Presentations (10 pages total)

This notebook is but a shell and can be adapted and tailored to your curriculum. The name, though cheesy, is a nod to the philosophical nature of logic and fostering the shift from
computation to abstract logic. This is a place where students discover and reflect upon logic and how it fits into mathematics, philosophy, and their lives as a whole.

These exercises throughout the pedagogical pathways are meant to supplement a strong curriculum and build a bridge, board by board, into mathematical logic using current educational research and instructional shifts.
4.2 It’s All Greek to Me: Symbolism and the Foundations of Mathematical Logic
Pre-Unit Quiz

Abstract

The following assignment is a short pre-unit quiz to get students and professors on the same page about the course. It will address what students already know about the course, what they want to know, and where there major areas of interest lie. On the next page, there is a professor guide to organizing the information they receive from these quizzes in order to truly assess the results.

Objective

Students will use their previous knowledge to evaluate what they already know about logic, express what they expect from the course and indicate some areas they are particularly interested in.

Materials

Pre-Unit Quiz

Writing Utensils

Overview

This quiz should be given to students at the very beginning of the course or unit on logic. There is no right or wrong answer and students should take it seriously, but not be nervous about answering wrong. Make sure that students know that this is an opportunity for them to take responsibility for their own learning. Their feedback is heard, considered, and valued. This is an opportunity to open the lines of communication between professor and student.
Pre-Unit Quiz

Answer the following questions honestly and completely. There is no wrong answer.

1. What is your Classification, Major, and reason for taking this course? ________________

2. In your own words define Mathematical Logic: ________________________________

3. What do you expect to study/cover in this course? ________________________________

4. Do you think we use logic every day? If yes, where? ______________________________

5. Using your previous knowledge of math and logic, do you think they intersect? If yes, in what ways? ________________________________

6. In your opinion, is proof necessary to mathematics? Why? _________________________

In the following categories please indicate your interest level:

<table>
<thead>
<tr>
<th>Category</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>History</td>
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<tr>
<td>Philosophy</td>
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<tr>
<td>Law/Criminal Justice</td>
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<td>Computer Science</td>
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<tr>
<td>Geometry</td>
<td></td>
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</tbody>
</table>
Guide for Organizing Feedback

Course: ___________________________  Semester: ___________________________

1. Classification and Major of Study Demographics

<table>
<thead>
<tr>
<th>Major</th>
<th>Number of Students</th>
</tr>
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<tbody>
<tr>
<td></td>
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</tbody>
</table>

2. Overall, do students know the definition of Discrete Mathematics?

   1  2  3  4  5

   Comments: ____________________________________________

3. Were there any surprising responses in reference to what students expect to cover in this course?

   ______________________________________________________
   ______________________________________________________
   ______________________________________________________

   What can I do in response to this feedback? ____________________________
   ______________________________________________________
4. Do the students have a grasp of logic, its uses and its importance to mathematics?

   1  2  3  4  5

   Comments: ____________________________________________
   _______________________________________________________
   _______________________________________________________
   _______________________________________________________
   _______________________________________________________

5. **Average Interest Levels**

   History.................................................. 1  2  3  4  5
   Philosophy............................................ 1  2  3  4  5
   Law/Criminal Justice............................... 1  2  3  4  5
   Computer Science................................. 1  2  3  4  5
   Geometry............................................. 1  2  3  4  5

   Comments: ____________________________________________
   _______________________________________________________
   _______________________________________________________
   _______________________________________________________
   _______________________________________________________

6. **Additional Notes:**

   _______________________________________________________
   _______________________________________________________
   _______________________________________________________
   _______________________________________________________
   _______________________________________________________

   53
Vocabulary Mapping – Basic Terminology

To be included in Notebook

Abstract

The following vocabulary mapping is not only meant to introduce students to each new concept and term but also reveal the interconnection of the vocabulary with each other.

Objective

Students will organize the vocabulary introduced in basic logic to gain knowledge and chart the interconnection of each term.

Materials

Blank Vocabulary Mapping

Writing Utensil

Overview

These worksheets should be given to students at the beginning of the course/unit on mathematical logic to help them organize the new vocabulary and data. It is designed to aid in the progression from one term to another and also each one offers a place to note the definition, any synonyms, the concept using symbols, and the concept in words. This should be included in the students’ progressive and cumulative notebook throughout the semester therefore no grading rubric is included. I have included a blank copy of the map and a completed map. NOTE: The following two premises are used in the connectors portions: p: It is raining outside, q: I take an umbrella.
Vocabulary Mapping – Basic Terminology

Every Proof Starts with:

- **Synonyms**
- **Statements**
- **Definition**
- **Notation:**
- **Primitive Statements**
- **Conclusions**
- **Definition**

These statements are combined into **Compound Statements**
These statements become **Compound Statements** when they are combined using connectors.

**Four Common Connectors**

- **Biconditional**
- **Implication**
- **Conjunction**
- **Disjunction**

**Additional Note:**

The Exclusive *OR*
These **Compound Statements** are evaluated using:

**Truth Tables**

**Truth Value**

**Tautology**

**Contradiction**

**Valid Argument**

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P ∨ Q</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>
Vocabulary Mapping SAMPLE KEY

Every Proof Starts with:

**Statements**
- **Synonyms**
  - Proposition
  - Premise

**Notation:**
Commonly denoted by letters of the alphabet

**Definition**
A declarative sentence that is either true or false.

**Primitive Statements**

**Definition**
A statement that cannot be broken down or simplified, it is in its most basic form.

**Conclusions**

**Definition**
A conclusion is the statement that you are attempting to prove in an argument. This statement should be presented at the beginning of the proof and then the proof should show that it logically proceeds from given/accepted premises.

These statements are combined into **Compound Statements**
These statements become **Compound Statements** when they are combined using connectors.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Sample Sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \leftrightarrow q )</td>
<td>Read “( p ) if and only if ( q )”. ( p \leftrightarrow q ) occurs whenever ( p ) is necessary and sufficient for ( q ).</td>
<td>It is raining if and only if I take an umbrella.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Sample Sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \rightarrow q )</td>
<td>Read “( p ) implies ( q )” or “If ( p ) then ( q )”. NOTE: There doesn’t need to be a causal relationship between ( p ) and ( q ) for the implication to be true.</td>
<td>If it is raining then I take an umbrella. NOTE: It is possible to take an umbrella whenever it is not raining. Also ‘if it is not raining then I do not take an umbrella’ also makes sense.</td>
</tr>
</tbody>
</table>

### Four Logical Connectors

- **Biconditional**: \( p \leftrightarrow q \)
  - Read “\( p \) if and only if \( q \)”. \( p \leftrightarrow q \) occurs whenever \( p \) is necessary and sufficient for \( q \).
  - Sample Sentence: It is raining if and only if I take an umbrella.

- **Implication**: \( p \rightarrow q \)
  - Sample Sentence: If it is raining then I take an umbrella. NOTE: It is possible to take an umbrella whenever it is not raining. Also ‘if it is not raining then I do not take an umbrella’ also makes sense.

- **Conjunction**: \( p \land q \)
  - Read “\( p \) AND \( q \)”. The statement \( p \land q \) is true if and only if \( p \) is true and \( q \) is true.
  - Sample Sentence: It is raining outside AND I take an umbrella

- **Disjunction**: \( p \lor q \)
  - Read “\( p \) OR \( q \)”. The statement \( p \lor q \) is true if and only if \( p \) or \( q \) is true, BOTH \( p \) and \( q \) are true.
  - Sample Sentence: 1. It is raining outside and I don’t take an umbrella 2. It is not raining outside and I take an umbrella 3. It is raining outside and I take an umbrella

**Additional Note:**

The **Exclusive OR**

Denoted by: \( p \nleftrightarrow q \)

The exclusive or occurs if and only if \( p \) OR \( q \) is true, but not both.

1. It is raining and I don’t take an umbrella.
2. It is not raining and I take an umbrella.
These **Compound Statements** are evaluated using:

### Truth Tables

**Definition:** A table in which the truth values of compound statements are evaluated and summarized.

**Example:**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Q</td>
<td>P ∨ Q</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>0</td>
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<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Truth Value

**Definition:** A statement has truth value whenever it is true or false, its truth or falsity can be investigated.

**Example:** What a delightful movie has no truth value because it is an opinion. $3 + 1 = 4$ does have truth value, it is true.

### Tautology

**Definition:** Tautology occurs whenever a statement is true for all truth values.

**Example:** $p ∨ ¬p$ is a tautology. It is raining or it is not raining. This is always true.

### Contradiction

**Definition:** A statement is considered a contradiction if it is false for all truth values.

**Example:** $p ∧ ¬p$ is a contradiction. It is raining and not raining.

### Valid Argument

**Definition:** A valid argument is a compound statement where when the given premise(s) are true the implication of the conclusion is also true (tautology).

**Example:** ‘If it is raining then I bring an umbrella’ is a valid argument.
Sudoku Puzzle Logic

Abstract

In this investigative exercise, students will use something familiar to most, a basic Sudoku puzzle, to investigate the logic of a Sudoku puzzle, whether or not a statement pertaining to that square is correct, and the rationale behind it.

Objective

Students will investigate the logical properties of a Sudoku puzzle. They will evaluate truth statements and implication statements pertaining to the puzzle.

Materials

Blank Sudoku Puzzle Worksheet

Overview

This exercise should be given to students shortly after introducing them to truth values and implication statements. It connects an everyday activity of solving a simple Sudoku puzzle to mathematical logic. Students will have to put their logical think into written form. The will help them start working through each step of their reasoning in writing which is vital to writing proofs. This assignment is an investigative activity; it can be done in groups or individually. Students must have background knowledge of Sudoku puzzles in order to complete the assignment. Depending on the students, a short summary of Sudoku puzzles before the lesson may be needed for students who are not familiar with them.
Sudoku Puzzle Logic

A word on Sudoku puzzles: A Sudoku puzzle has the number 1-9 in every row and column of the puzzle as well as every bolded square. Each number occurs only once in each and cannot be repeated.

Analyze the following statements using the puzzle above. Each rationale should be answered completely and in sentence form. Every statement you make (including those in your rationale) should be verifiable through primitive statements or already accepted statements.

EXAMPLE: Statement: H1 is the number 5

<table>
<thead>
<tr>
<th>Truth Value:</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rationale:</td>
<td>The number 5 can only occur once per row.</td>
</tr>
<tr>
<td></td>
<td>F1 is the number 5; So column 5 already contains a two.</td>
</tr>
<tr>
<td></td>
<td>Therefore H1 cannot be the number 5</td>
</tr>
</tbody>
</table>

1) Statement: E3 is the number 2

<table>
<thead>
<tr>
<th>Truth Value:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rationale:</td>
<td></td>
</tr>
</tbody>
</table>
2) What is E3?
   Write a statement and explain your rationale in reference to this value.

<table>
<thead>
<tr>
<th>Statement:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rationale:</td>
<td></td>
</tr>
</tbody>
</table>

3) **Statement:** C8 is 3

<table>
<thead>
<tr>
<th>Truth Value:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rationale:</td>
<td></td>
</tr>
</tbody>
</table>

4) **Statement:** If H7 is 5 then H9 is the number 8

<table>
<thead>
<tr>
<th>Truth Value:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rationale:</td>
<td>Assume that H7 is 5.</td>
</tr>
</tbody>
</table>

5) **Statement:** If I2 is 4 then B1 is 4 OR C1 is 4

<table>
<thead>
<tr>
<th>Truth Value:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rationale:</td>
<td></td>
</tr>
</tbody>
</table>
6) Solve the remaining puzzle and write statements for your next 4 moves including the rationale behind each step.

   a. Statement: ______________________________
      i. Rationale: ______________________________
      _______________________________________
      _______________________________________
      _______________________________________
      _______________________________________

   b. Statement: ______________________________
      i. Rationale: ______________________________
      _______________________________________
      _______________________________________
      _______________________________________
      _______________________________________

   c. Statement: ______________________________
      i. Rationale: ______________________________
      _______________________________________
      _______________________________________
      _______________________________________
      _______________________________________

   d. Statement: ______________________________
      i. Rationale: ______________________________
      _______________________________________
      _______________________________________
      _______________________________________
      _______________________________________
Sudoku Puzzle Logic: SAMPLE KEY

A word on Sudoku puzzles: A Sudoku puzzle has the number 1-9 in every row and column of the puzzle as well as every bolded square. Each number occurs only once in each and cannot be repeated.

Analyze the following statements using the puzzle above. Each rationale should be answered completely and in sentence form. Every statement you make (including those in your rationale) should be verifiable through primitive statements or already accepted statements.

EXAMPLE: Statement: H1 is the number 5

<table>
<thead>
<tr>
<th>Truth Value: 0</th>
</tr>
</thead>
</table>
| **Rationale:** The number 5 can only occur once per row.  
  F1 is the number 5; So column 5 already contains a two.  
  Therefore H1 cannot be the number 5 |

1) Statement: E3 is the number 2

<table>
<thead>
<tr>
<th>Truth Value: 0</th>
</tr>
</thead>
</table>
| **Rationale:** The number 2 can only occur once per row.  
  A3 is the number 2; so column 3 already contains the number 2.  
  Therefore E3 cannot be 2. |
2) **What is E3?**

   Write a statement and explain your rationale in reference to this value.

<table>
<thead>
<tr>
<th><strong>Statement:</strong></th>
<th>E3 is the number 8</th>
</tr>
</thead>
</table>
| **Rationale:** | Columns 1 and 2 contain the number 8.  
                  Row D also contains the number 8.  
                  Therefore E3 must be 8. |

3) **Statement:** C8 is 3

<table>
<thead>
<tr>
<th><strong>Truth Value:</strong></th>
<th>1</th>
</tr>
</thead>
</table>
| **Rationale:**   | Rows A and B contain the number 3.  
                  Therefore C8 must be 3. |

4) **Statement:** If H7 is 5 then H9 is the number 8

<table>
<thead>
<tr>
<th><strong>Truth Value:</strong></th>
<th>1</th>
</tr>
</thead>
</table>
| **Rationale:**   | Assume that H7 is 5.  
                  Rows G and I already contain the number 8.  
                  Therefore either H7 or H9 must be 8.  
                  Since H7 is 5 then H9 must be 8. |

5) **Statement:** If I2 is 4 then B1 is 4 OR C1 is 4

<table>
<thead>
<tr>
<th><strong>Truth Value:</strong></th>
<th>1</th>
</tr>
</thead>
</table>
| **Rationale:**   | Assume that I2 is 4.  
                  Columns 2 and 3 contain the number four.  
                  The square formed by Rows A-C and Columns 1-3 must contain the number four,  
                  Therefore B1 or C1 must contain the number 4. |
6) Solve the remaining puzzle and write statements for your next 4 moves including the rationale behind each step.

   a. Statement: F7 is 3
      i. Rationale: Columns 8 and 9 already contain the number 3
                     Row E also contains the number 3
                     Therefore F3 must be 3
                     
   b. Statement: H4 is 3
      i. Rationale: Rows G and I contain 3
                     Columns 5 and 6 contain 3
                     Therefore H4 must be 3
                     
   c. Statement: F8 is the number 8
      i. Rationale: Rows D and E both contain the number 8
                     Column 9 also contains 8
                     Therefore F8 must be 8
                     
   d. Statement: A7 is the number 8
      i. Rationale: Columns 8 and 9 both contain 8
                     Row B also contains 8
                     Therefore A7 must be 8
                     

Journal, Reflections, and Ponderings

To be included in notebook

Abstract

The material presented in Discrete Mathematics requires the utmost critical thinking skills. If students never stop to connect the material together, wander aimlessly throughout the course and ‘miss the forest for the trees’. The following questions and reflection aim to give students a moment to step back, connect concepts, and analyze the big picture. This portion of the notebook is also a place where students practice concepts and it presents the students and professor with a collaboration opportunity.

Objective

Students will use their critical thinking skills to reflect upon concepts and questions dealing with mathematical logic and the philosophy of mathematics. They will also be given an opportunity to present any additional ponderings for collaborative review as well as practice concepts in a non-traditional way.

Material

Blank Journal Prompt and Writing Utensil

Overview

The journal reflections and ponderings section should be an entire tab of the notebook. It is a portion built over time, students should receive these questions throughout the course to promote critical thinking and solidify concepts in a student’s mind. Their aim is to ask students tough questions that they may have never considered, it also provides a place for students to practice and deeply consider/analyze concepts presented to them in discrete mathematics. The format is as follows: Students will be presented with a creative assignment or critical thinking question to reflect upon within the ‘journal and reflections’ portion of the page. They will also be presented with a quote from an early mathematician or philosopher on which to expand and give their thoughts. This will be done in the ‘additional ponderings’. The feedback on the right side of the page is for any
comments or questions the professor would like to offer in response to the student’s work. The journal, reflections, and ponderings section of the notebook an opportunity for students to, as Shakespeare once said, “Work you thoughts” and aids in creating a critical learning environment.

There are journals for each section of this paper (*Logical Symbolism, The Truth about Truth Tables, The Laws of Logic, and Constructing a Proof all Your Own*). They are meant to enrich that topic. Each journal entry will be included within that section of the Pedagogical Pathways and also in italics underneath the heading of the journal entry.
Journal and Reflections

*It’s All Greek to Me: Symbolism and The Foundations of Mathematical Logic*

Write an instructional How-to guide of your favorite hobby (i.e., *How to Swing a Baseball Bat* or *How to Bake a loaf of Bread*). This should be written in bulleted form and you must use at least 5 connectors as well as 2 implication (if-then) statements within your document. (Example: If the bread is doughy \(\rightarrow\) add flour)

---

**Additional Ponderings: Reflect on this quote:**

“*Mathematics is so rich that no form of representation can hope to capture all of it. Curves actually have the property of degree, which we stumble upon only by inventing the algebraic notation of analytical geometry. Knots – though you-de never know it from playing with bits of string – have properties associated with continued fractions and polynomials. And we discover these properties only by inventing new notations which make them manifest.*” James Robert Brown – Philosophy of Mathematics (1999)
### Journal and Reflections

**It’s All Greek to Me: Symbolism and The Foundations of Mathematical Logic**

Write an instructional How-to guide of your favorite hobby (ie, *How to Swing a Baseball Bat* or *How to Bake a loaf of Bread*). This should be written in bulleted form and you must use at least 5 connectors as well as 2 implication (if-then) statements within your document. (Example: If the bread is doughy $\Rightarrow$ add flour)

**How to Make Peppermint Bark**

**You will need:**
- White Chocolate Almond Bark
- Semi-Sweet Chocolate
- Peppermint Extract
- Crushed Candy Canes

- Grease a sheet pan $\lor$ Melt the Semi-Sweet Chocolate in a double boiler.
- When it is smooth $\lor$ creamy stir in the peppermint extract.
- If you like more peppermint $\Rightarrow$ add extra.
- Pour the chocolate on a greased sheet pan $\lor$ place it in the fridge to cool.
- Melt the white chocolate bark in a double boiler $\land$ microwave
- Stir in crushed candy canes.
- NOTE: If you want to save some to sprinkle on top $\Rightarrow$ set a small pile aside
- Take Semi-sweet chocolate out of the fridge $\lor$ spread the white chocolate on top.
- Sprinkle on candy canes if you desire.

**Additional Ponderings: Reflect on this quote:**

“*Mathematics is so rich that no form of representation can hope to capture all of it. Curves actually have the property of degree, which we stumble upon only by inventing the algebraic notation of analytical geometry. Knots – though you-de never know it from playing with bits of string – have properties associated with continued fractions and polynomials. And we discover these properties only by inventing new notations which make them manifest.*” James Robert Brown – Philosophy of Mathematics (1999)

Math is a great and vast subject. It not only forms the framework for many of the sciences today but also is vital to everyday life. It is not however a closed subject, there are many avenues and roads not yet traveled or paved yet. The key to these roads lies within me, the future of mathematics lies within each of us.
Journal and Reflections

It's All Greek to Me: Symbolism and The Foundations of Mathematical Logic

You are a newspaper columnist, a reader writes you with the following questions.

To Whom it May Concern:

Mathematics is something I have been taught since elementary school. I know that 2+2 = 4 and other basic computation rules but no one has ever answered the questions: What is mathematics outside of computation? Also, how do we know that things like the Pythagorean Theorem work for all triangles in the universe?

Sincerely,
Curious and Confused in Chicago

Write your response below:

________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________

Additional Ponderings: Reflect on this quote:

“Mathematics is so rich that no form of representation can hope to capture all of it. Curves actually have the property of degree, which we stumble upon only by inventing the algebraic notation of analytical geometry. Knots – though you-de never know it from playing with bits of string – have properties associated with continued fractions and polynomials. And we discover these properties only by inventing new notations which make them manifest.” James Robert Brown – Philosophy of Mathematics (1999)
Journal and Reflections

It’s All Greek to Me: Symbolism and The Foundations of Mathematical Logic

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Sincerely,
Curious and Confused in Chicago

Write your response below:

Dear Curious and Confused,

Thank you for your submission. What you have asked is a very abstract question. Math is something that describes all of the patterns within our universe, or attempts to. There are many math avenues which have not been perfected or completed. There was no clear beginning to math and there is no end. It is something we know so little about but yet something inside of us, sort of an intuition, knows when something is mathematically out of proportion. It is also consistent through proofs. We can’t possibly draw every triangle in the universe, there is infinitely many but through logic and accepted statements and premises we can proof with ease that the Pythagorean Theorem works for all right triangles.

Kind Regards,

Discrete Math Dominator in Delaware

Additional Ponderings: Reflect on this quote

“A Philosopher who has nothing to do with geometry is only have a philosopher and a mathematician with no element of philosophy in him is only half a mathematician. These disciplines have estranged themselves from one another to the detriment of both.” –Frege

There is an obvious intersection between math and logic, there has to be a structure to govern the laws of mathematics or we would never get anywhere and no one would know what is valid and what is not. Knowledge of the axiomatic method is vital in proof building as well as philosophical demonstration they are peas in the same pod.
4.3 The Truth about Truth Tables
Vocabulary Mapping – Conjugate Propositions

To be included in Notebook

Abstract

The following vocabulary/concept map is designed to help students with the difference between contrapositive, inverse, and converse of a conditional statement as well as explore their relationship to one another.

Objective

Students will organize and explore contrapositive, inverse, and converse of a conditional statement and how they interact.

Materials

Blank Vocabulary Mapping

Overview

This worksheet should be given to student at the introduction of conjugate propositions. Converse, Contrapositive, and Inverse are not only similar in looking and sounding they are also similar in meaning but can produce vastly different results. Students recognizing the differences and similarities between these concepts can aid them greatly whenever building proofs. It will not only help them avoid some common pitfalls of proof building such as assuming that the inverse of a conditional statement is true because the original implication statement is true, but also help with the laws of logic, such as modus tollens: 

\[ (P \to Q) \land \neg Q \]

This worksheet presents a truth table so students discover the rules through what they already know about truth tables and helps visually aid the students in this tricky terminology.
Vocabulary Mapping – Conjugate Propositions

Conditional Proposition

- **p**: x is a positive number
- **q**: \( x^2 \) is a positive number

\[ p \rightarrow q \]

Statement form:

Converse

Definition:

Statement Form:

Contrapositive

Definition:

Statement form:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( p \rightarrow q )</th>
<th>( q \rightarrow p )</th>
<th>( \neg q \rightarrow \neg p )</th>
<th>( \neg p \rightarrow \neg q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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Inverse

Definition:

Statement Form:
Vocabulary Mapping – Conjugate Propositions: SAMPLE KEY

Conditional Proposition

\( p: x \) is a positive number
\( q: x^2 \) is a positive number

\[ p \rightarrow q \]

Statement form:

If \( x \) is a positive number then \( x^2 \) is a positive number

Contrapositive

Obtained by:

Interchanging the antecedent and the consequent and replacing them with their negation. It is the converse of the inverse.

Statement form:

If \( x^2 \) is not a positive number then \( x \) is not a positive number

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Converse

Obtained by:

The consequence of interchanging the antecedent and the consequent in the original proposition

Statement Form:

If \( x^2 \) is a positive number then \( x \) is a positive number

Inverse

Obtained by:

Replacing the antecedent and the consequent with their negation

Statement Form:

If \( x \) is not a positive number then \( x^2 \) is not a positive number
The Constitution of Logic

To be included in the Notebook

Abstract

In this assignment students work in collaborative groups and will be given a proposed law and 4 compound statements and they will determine their equivalence relationship and if a law exists among the statements. They will then write the law in their own words and prove their support using truth tables and sentence statements.

Objective

Students will collaboratively use what they already know about truth tables and logical equivalence to discover the laws of logic and then present them to their classmates.

Materials

Blank Laws of Logic Worksheet

Computer with Microsoft PowerPoint and a Projector

Overview

Knowing the Laws of Logic is imperative for student’s success in proof building. They are the very foundation for student’s simplifying and connecting statements. They also help student’s avoid some common logical mistakes in proof building. This assignment is designed so that students work through the laws of logic step by step. They are given a blank truth table with only the compound statements filled in and then they must work through the truth table and determine any relationship that those statements have with one another. Namely, whatever law they are working on. They will then write the law in symbols, using formal language, and then in their own words including any tricks they thought of or pneumonic devices to help them remember the law. They will then present that to the class which constructs the laws of logic
constitution. The class should be split up into cooperative groups and each group given one law worksheet to analyze and present. Part of this worksheet is to help students realize the trickiness of written language and the necessity of logical notation. It also presents the students an opportunity to practice their skills in truth table building and evaluating. Before each group presents, all students should be given one of each worksheet so that while students are presenting they can fill in their own constitution.

The Laws of Logic are split into two sections, the Laws of Equivalence and the Rules of Inference. In the first section, the Laws of Equivalence because of the nature of the truth tables that correspond with these laws; the Idempotent Laws, Identity Laws, Inverse Laws, and Domination Laws may need to be paired up within one group. Their truth tables are fairly similar and not as involved as some of the others. This forms one entire section of the notebook. I have pre-worked the law of double negatives to be given to each student as an example.
The Law of Double Negatives

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Law Written in Symbols: \( \neg \neg p \Leftrightarrow p \)

Law Written in Formal Language:

*Negating an already negated statement is logically equivalent to the original statement*.

Law Written in Our Own Words:

*Just like in algebra: \(-(-x) = x\). Or in English the statement: “I am not not going to the store” is a double negative and implies that I am going to the store. In logic a negating a not statement yields the original statement.*
The Laws of Logic

DeMorgan’s Laws

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Law Written in Formal Language:

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Law Written in Our Own Words:

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Law Written in symbols: \( \neg (p \land q) \leftrightarrow \neg p \lor \neg q \)  
\( \land \)  
\( \neg (p \lor q) \leftrightarrow \neg p \land \neg q \)

Law Written in Formal Language:

Negating a conjunction statement is logically equivalent to the disjunction of the negations.

Negating a disjunction statement is logically equivalent to the conjunction of the negations.

Law Written in Our Own Words:

Similar to algebra, this law deals with distribution of negation over disjunction and conjunction statements. Whenever you have the negation of a disjunction or conjunction statement you distribute it to each term just like in algebra and then flip your conjunction to a disjunction or vice versa.
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Law Written in Formal Language:  
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Law Written in Our Own Words:  
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The Laws of Logic

Commutative Laws
The Laws of Logic

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Law Written in Symbols: $p \lor q \Leftrightarrow q \lor p$

AND $p \land q \Leftrightarrow q \land p$

Law Written in Formal Language:

A disjunction statement is logically equivalent even if the statements are reversed.

A conjunction statement is logically equivalent even if the statements are reversed.

Law Written in Our Own Words:

Just like in addition where $a + b = b + c$, disjunction and conjunction statements are commutative. In English “I am going to the store and eating dinner out” is the same as “I am eating dinner out and going to the store”. Also “I am going to the store or eating dinner out” is the same as “I am eating dinner out or going to the store”.

84
### The Laws of Logic

#### Associative Laws

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**Law Written in Symbols:**

**Law Written in Formal Language:**

**Law Written in Our Own Words:**
The Laws of Logic

### Associative Laws

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**Law Written in Symbols:**

\[ p ∨ (q ∨ r) \Leftrightarrow (p ∨ q) ∨ r \]

AND \[ p ∧ (q ∧ r) \Leftrightarrow (p ∧ q) ∧ r \]

**Law Written in Formal Language:**

A multiple statement disjunction statement is logically equivalent no matter the order of disjunction.

A multiple statement conjunction statement is logically equivalent no matter the order of conjunction.

**Law Written in Our Own Words:**

Just as in algebra, \( a + (b + c) = (a + b) + c \), in logic disjunction and conjunction statements are associative. Also in English: I am going to the store to buy milk and bread and butter is equivalent to I am going to the store to buy bread and butter and milk.
## The Laws of Logic

### Distributive Laws

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**Law Written in Symbols:**

- $p \land (q \lor r) = (p \land q) \lor (p \land r)$
- $(p \lor q) \land (p \lor r) = p \lor (q \lor r)$

**Law Written in Formal Language:**

- Distributive Law: $p \land (q \lor r) = (p \land q) \lor (p \land r)$
- Distributive Law: $(p \lor q) \land (p \lor r) = p \lor (q \lor r)$

**Law Written in Our Own Words:**

- The distributive laws state that $p \land (q \lor r) = (p \land q) \lor (p \land r)$ and $(p \lor q) \land (p \lor r) = p \lor (q \lor r)$.
The Laws of Logic

### Distributive Laws

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Law Written in Symbols:  
\[ p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r) \quad \text{AND} \quad p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r) \]

Law Written in Formal Language:

*The conjunction of a disjunction statement is logically equivalent to the disjunction of the distribution of the conjunction to each statement.*

*The disjunction of a conjunction statement is logically equivalent to the conjunction of the distribution of the disjunction to each statement.*

Law Written in Our Own Words:

*Just like distribution in algebra: \( a(b+c) = ab + bc \), you can distribute a conjunction across a disjunction or a disjunction across a conjunction.*
Idempotent Laws

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Law Written in Symbols: 

Law Written in Formal Language:

Law Written in Our Own Words:
The Laws of Logic

Idempotent Laws

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<td>$p$</td>
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Law Written in Symbols:

$p \lor p \Leftrightarrow p$

$p \land p \Leftrightarrow p$

Law Written in Formal Language:

A conjunction of a statement with an identical statement is logically equivalent to the statement.

A disjunction of a statement with an identical statement is logically equivalent to the statement.

Law Written in Our Own Words:

Just like in English to say the statement: “I am going to the store or I am going to the store” is redundant and is the same as saying “I am going to the store”. It is also true using and: I am going to the store and I am going to the store” is equivalent to “I am going to the store”.

90
### Identity Laws

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<th>$F_0$</th>
<th>$T_0$</th>
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<th>$p \land T_0$</th>
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**Law Written in Symbols:**

**Law Written in Formal Language:**

**Law Written in Our Own Words:**
The Laws of Logic

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Law Written in Symbols: \(p \lor F_0 \iff p\) AND \(p \land T_0 \iff p\)

Law Written in Formal Language:

The disjunction of a statement and a contradiction is logically equivalent to the statement.

The conjunction of a statement and a tautology is logically equivalent to the statement.

Law Written in Our Own Words:

In English to combine an absurd statement with a true statement using or is logically equivalent to the true statement. For example: "I am going to the store or I live on Mars" is the same as saying "I am going to the store". The tautology statement is a bit more tricky, it would be like combining "I am going to the store AND I live on Earth". Well if you are reading this paper you live on Earth, so just saying "I am going to the store" is logically equivalent.
The Laws of Logic

Inverse Laws

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<tr>
<td>$p$</td>
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<td>$p \lor \neg p$</td>
<td>$p \land \neg p$</td>
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Law Written in Symbols: __________________________________________

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Law Written in Formal Language:

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Law Written in Our Own Words:

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The Laws of Logic

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Law Written in Symbols:

\[ p \lor \neg p \Leftrightarrow T_0 \quad \text{AND} \quad p \land \neg p \Leftrightarrow F_0 \]

Law Written in Formal Language:

The disjunction of a statement and its negation is tautology.

The contradiction of a statement and its negation is a contradiction.

Law Written in Our Own Words:

This law can be easily paralleled with English statements. For Example: The statements “I have brown eyes or I do not have brown eyes” is always true. My eyes are either brown or not. The statement I have brown eyes and I do not have brown eyes is a contradiction. How can a person’s eyes be brown but not brown simultaneously?
The Laws of Logic

### Domination Laws

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<th>$F_0$</th>
<th>$p \lor T_0$</th>
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**Law Written in Symbols:**

**Law Written in Formal Language:**

**Law Written in Our Own Words:**
## The Laws of Logic

**Domination Laws**

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<th>$p$</th>
<th>$T_0$</th>
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<th>$p \lor T_0$</th>
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**Law Written in Symbols:**

\[ p \lor T_0 \Leftrightarrow T_0 \]

**AND**

\[ p \land F_0 \Leftrightarrow F_0 \]

**Law Written in Formal Language:**

The disjunction of a statement and tautology is tautology.

The conjunction of a statement and a contradiction is a contradiction.

**Law Written in Our Own Words:**

This can be paralleled to English statements. For example: “I am 12 feet tall or I live on Earth” is a true statement because I live on Earth is always a true statement regardless of whether or not I am 12 ft. tall. The statements “I am 12 ft. tall and I live on Earth” is always a contradiction and never true regardless of whether or not I live on Earth, I am not 12 ft. tall.
Absorption Laws

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Law Written in Symbols: ____________________________

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Law Written in Formal Language:

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Law Written in Our Own Words:

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Absorption Laws

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</table>

Law Written in Symbols: \( p \land (p \lor q) \Leftrightarrow p \) AND \( p \lor (p \land q) \Leftrightarrow p \)

Law Written in Formal Language:

The conjunction of a statement with a compound disjunction statement containing the original statement is logically equivalent to the original statement.

The disjunction of a statement and a compound conjunction statement containing the original statement is logically equivalent to the original statement.

Law Written in Our Own Words:

This law can be paralleled with English using the statements: \( p \): It is raining and \( q \): I have an umbrella. Consider this sentence, pardon the redundancy “It is raining AND it is raining or I have an umbrella”. No matter if I have an umbrella or not, it is raining. Also consider the statement “It is raining OR it is raining and I have an umbrella” either way it is raining.
STOP! In the Name of Truth Tables

Abstract

In this research-based assignment students will find an instructional sign of their choosing and create a truth table of its statements. This includes but is not limited to No Parking signs, street signs of any kind, and signs in the doors of restaurants in reference to their hours or store policies.

Objective

Students will evaluate instructional signs in their surroundings using their knowledge of truth tables.

Materials

Blank STOP! In the Name of Truth Tables with Assignment Page and a writing utensil

Camera (optional)

Overview

This assignment is aimed at students who are studying truth tables and their truth values. It is aimed to help students connect symbols and tables of mathematical jargon into real world situations to aid in their learning of logical processing. Students will be instructed to find an instructional sign, sketch it (or photograph it), and create a truth table of its statements. For example a sign that reads “No Parking: All offenders will be towed at owner’s expense” could be evaluated in a truth table of 2 statements: $p$, and $q$. Where $p$: You are parked in the no parking zone, $q$: Your car will be towed (see assignment for truth table). Students are allowed to take charge of their education and encouraged to challenge their creativity in this assignment.
STOP! In the Name of Truth Tables

Truth tables are tables of logical consequences from given premises. They are used in discrete mathematics but also hook into everyday life in an organized non-ambiguous way that language cannot accomplish in some instances. Consider the street sign “No Parking, All offenders will be towed at owner’s expense”. This sign can be organized into a two statement truth table using the statements $p$: You are parked in the no parking zone, and $q$: your car will be towed.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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<tbody>
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The Truth Table of “No Parking, All Offenders will be towed at Owner’s Expense”

Your Assignment:

Your assignment is to find an instructional sign with at least 3 statements (for example “No Shoes, No Shirt, No Service”) and create a truth table analyzing the statements. This could be but is not limited to: a street sign, a sign posted in the door of a restaurant or establishment, etc. Your final page must meet the following requirements:

- A sketch or photograph of the sign of your choosing and its location.
- Your 3 statements clearly labeled (ie. $p$: you are parked in the no parking zone).
- A truth table analyzing the statements.
Sign Used: “No Shoes, No Shirt, No Service”

Location: Speedy Mart on Gumdrop Lane

Sketch or Picture:

![NO Shoes
NO Shirt
NO Service]

Statements:

p: You will receive service
q: You are wearing a shirt
r: You are wearing shoes

Truth Table:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>\neg p</th>
<th>\neg q</th>
<th>\neg r</th>
<th>p \land q</th>
<th>\neg(p \land q) = \neg q \lor \neg r</th>
<th>(\neg q \lor \neg r) \rightarrow \neg p</th>
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</table>
Journal and Reflections

The Truth about Truth Tables

Solve the puzzle below. Use the table given as a guide to organizing the information.

Smith, John, and Tim live in Houston, Dallas, and Ft. Worth, not necessarily in that order. One particular year they all travel out of town for the holidays. Smith and John travel to Ft. Worth for the holidays. Tim travels to Dallas for the Holidays. Smith lives further south that Tim. Where does John live?

<table>
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<tr>
<th></th>
<th>Dallas</th>
<th>Ft. Worth</th>
<th>Houston</th>
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</thead>
<tbody>
<tr>
<td>Smith</td>
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<tr>
<td>John</td>
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<tr>
<td>Tim</td>
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Feedback

Additional Ponderings: Reflect on this quote:

“It seems to me that no philosophy can possibly be sympathetic to the mathematician which does not admit, in one matter or another, the immutable and unconditional validity of mathematical truth. Mathematical Theorems are true or false, their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality.” —G.H. Hardy (1929)
Journal and Reflections
The Truth about Truth Tables

Solve the puzzle below. Use the table given as a guide to organizing the information.

Smith, John, and Tim live in Houston, Dallas, and Ft. Worth, not necessarily in that order. One particular year they all travel out of town for the holidays. Smith and John travel to Ft. Worth for the holidays. Tim travels to Dallas for the Holidays. Smith lives further south that Tim. Where does John live?

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<td>Smith</td>
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<tr>
<td>John</td>
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<tr>
<td>Tim</td>
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</table>

John Lives in Dallas

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Additional Ponderings: Reflect on this quote:

“It seems to me that no philosophy can possibly be sympathetic to the mathematician which does not admit, in one matter or another, the immutable and unconditional validity of mathematical truth. Mathematical Theorems are true or false, their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality.” –G.H. Hardy (1929)

G.H. Hardy is asserting that mathematical truth is everywhere, in everything, and true whether we acknowledge or discover it. This leaves room for mathematics that we have not yet discovered, pathways still untouched.

Mathematics is a vital part of every civilization and it binds us together

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_________________________________________________________
A peer of yours is confused about the two implications truth values, highlighted below. Explain in your own words why these occur and create your own sample statements to help them understand.

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<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
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**Sample Statements**

$p$: 
$q$: 

**Explanation:**

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**Additional Ponderings: Reflect on this quote:**

“Perhaps the best analogy is with baking a cake. If I’m successful, I can describe my non-linguistic activities of mixing the ingredients, setting the oven temperature, etc. in the form of a recipe, which of course, is a linguistic entity. But the recipe is not the activity of baking a cake; it is merely an aid to others who might want to bake a similar cake for themselves.” James Robert Brown *The Philosophy of Mathematics* (1999)
The Truth about Truth Tables

A peer of yours is confused about the two implications truth values, highlighted below. Explain in your own words why these occur and create your own sample statements to help them understand.

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</table>

Sample Statements

\( p \): It is raining

\( q \): I have an umbrella

**Explanation:**

1) If it is not raining then I have an umbrella. It is perfectly plausible to take an umbrella on a trip with you if rain is in the forecast or just in case.

2) If it is not raining then I don’t have an umbrella. On the same note, it is perfectly plausible to leave your umbrella at home if there is no rain in the forecast and it is sunny outside. Just beware that as this shows, a false premise can produce a false truth value.

Additional Ponderings: Reflect on this quote:

“Perhaps the best analogy is with baking a cake. If I’m successful, I can describe my non-linguistic activities of mixing the ingredients, setting the oven temperature, etc. in the form of a recipe, which of course, is a linguistic entity. But the recipe is not the activity of baking a cake; it is merely an aid to others who might want to bake a similar cake for themselves.” James Robert Brown *The Philosophy of Mathematics* (1999)

This is a perfect analogy for the aim of a mathematical proof. Math is not a linguistic entity, but a proof uses linguistics in many ways. A proof is a building block, if no one ever wrote down how to bake a cake and everyone who attempted had to recreate every step by trial and error we would never make advances. We must build on the recipes of others.
4.4 The Road Before us and the Road Behind: *History and Research Applications*
Vocabulary Mapping – Complete Concept Map

To be included in Notebook

Abstract

In previous maps students were given a format of how vocabulary naturally links together and their task was to fill in the map. In this map students must create their own unique map using a vocabulary list given.

Objective

Students will use their deep understanding of mathematical logic terminology to create a unique concept map from a list of vocabulary given.

Materials

Blank Concept Map Vocabulary List

Overview

It is easy for students to obtain surface knowledge of a subject, take a test, and then flush that knowledge out of your memory as soon as the test is over. This assignment challenges that mindset by giving students a list of vocabulary from multiple levels of mathematical logic and having them connect it into an interconnected web. Students must use their reasoning skills, sharpened through the study of logic, to create a logical chain of reasoning. Students will be given a list of vocabulary, they must arrange it in a logical way, connecting it with arrows and connector words such as ‘can be done by’, ‘explains’ or ‘involves’. Beforehand students need to have been introduced to all terms given, this list of vocabulary can of course be tailored to the particular course and curriculum. This assignment can be challenging to students at first because they must know a term at its core in order to connect it to another but if they work through it, they will come out with a deeper understanding of logical vocabulary.
Concept Map Assignment

A concept map is a web that’s aim is to interconnect vocabulary in a logical way. For example if I was given the terms: Emotion, Happy, Nervous, Angry, Sad, Tears, Frown, and Smile. I could arrange them as follows.

These major terms have been organized in a natural way using arrows and connection phrases.

Your task is to create a concept map from the logic terms given below. Every map will look different because everyone’s organization and understanding is a bit different. Every word must be used but connecting phrases are not necessary on every arrow.

<table>
<thead>
<tr>
<th>Concept Map Vocabulary:</th>
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<tbody>
<tr>
<td>• Proof</td>
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<tr>
<td>• Aristotle</td>
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<td>• Euclid</td>
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<tr>
<td>• ‘Lullism’</td>
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<tr>
<td>• Reasoning</td>
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<td>• Logic</td>
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<td>• Theorem</td>
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<td>• Contradiction</td>
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<td>• Premise</td>
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<tr>
<td>• Truth Table</td>
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<tr>
<td>• Compound Statements</td>
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<td>• Logical connectors</td>
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<td>• Disjunction</td>
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<td>• Conjunction</td>
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<td>• Negation</td>
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<td>• Tautology</td>
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<td>• Implication</td>
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<td>• Laws of Logic</td>
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<td>• Valid Argument</td>
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<td>• Lemma</td>
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<td>• Induction</td>
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<td>• Primitive statement</td>
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<td>• Hypothesis</td>
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</tbody>
</table>
Concept Map Assignment

- Assumption
- Primitive Statements
- Lullism
- Aristotle
- Euclid
- Logic
- Deduction
- Induction
- Contradiction
- Negation
- Proof
- Truth Tables
- Tautology
- Valid Argument
- Theorem
- Lemma

- Compound Statements
- Reasoning
- Premise
- Hypothesis
- Logical Connectors
- Four types
  - Disjunction
  - Conjunction
  - Implication
  - Biconditional

- Combined forms
- Starts with
- Used
- Evaluated using
- Philosophical
- Geometric
- Can be done by
- Can reveal
- Which infers a valid
- Must have a valid
The History of Mathematical Logic

Abstract

A deep understanding and knowledge of history is important to understanding the very foundations of mathematical logic. These History focuses assignments include building a profile of Euclid and Aristotle side by side containing their life and philosophies. The corresponding journal entry (located at the end of this section) proposes a merger between the two.

Objective

Students will step into the shoes of Aristotle and Euclid to compare and contrast their philosophies and write a persuasive letter proposing a merger between the two.

Materials

Blank Build-a-Profile

Blank Journal Assignment – to be included in notebook

Euclidean Proof Assignment

Overview

In order to fully grasp mathematical logic, students need to understand logic at its very basic core and beginnings. This is where the study of Aristotle and Euclid proves fruitful. Aristotle is known for contributing immensely to the foundations of logic, and Euclid to proof building with the axiomatic method. Though they operated in different fields, their logic systems were not far from each other as seen in the history portion of this paper. Through these assignments students will first build a profile of Euclid and Aristotle side by side. In the journal assignment they will write a professional letter to Euclid and
Aristotle proposing a merger between the two, giving evidence of their logical similarities and the benefits of combining the two styles.

The comparison worksheet should be given to students along with a history of logic lesson and filled in as the lecture progresses. This will not only help students stay engaged, but also allow them to study the core philosophies of these two great men side by side and draw their own conclusions about their similarities.

The journal is an opportunity for students to reflect upon these two philosophers and creatively express their similarities through a persuasive professional letter. Their challenge will be to write a letter Addressed to Euclid and Aristotle proposing a merger between their logic and how each one would benefit from a close partnership. Persuasion also uses logic, so they will have to hone their logic skills in order to present a well-rounded proposition.
A Comparative Profile of Euclid and Aristotle

<table>
<thead>
<tr>
<th></th>
<th>Euclid</th>
<th>Aristotle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Born</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Place of Birth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Death</td>
<td></td>
<td></td>
</tr>
<tr>
<td>School of Study</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Philosophy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Major Contributions</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notable Similarities:
A Comparative Profile of Euclid and Aristotle

Euclid

Lived: *From approximately 365 – 275 B.C.*

School of Study: Plato’s Academy or with students from Plato’s Academy

Philosophy: *Geometry can be explained starting with the most basic axioms, through deductive reasoning. “Lullism” and the axiomatic method, proving that results are correct through the most primitive statements and organizing them into logical structures.*

Major Contributions: *The Elements written around 300 B.C. that formed the foundation for geometry for thousands of years and still used today.*

Aristotle

Lived: *Approximately 385 – 322 B.C.*

School of Study: Plato’s Academy

Philosophy: *Syllogism and demonstrative Logic. A syllogism occurs whenever a conclusion is drawn from two or more given or assumed premises. A demonstration is an extended argument that begins with known premises and involves a chain of reasoning by deductively evident steps that its’ conclusion is a consequence of these premises.*

Major Contributions: *Credited with making major contributions to the formation of formal logic.*

Notable Similarities:

*Both studied at Plato’s academy which is said to have held the following slogan above its’ entrance “Let no one unversed in Geometry enter”*

*Both used known premises and logical deductive steps thereafter to prove statements.*
Computer Science Logic

Abstract

Computer Scientists and Programmers use logic every day. How else could they program a computer to play chess like a human? They have to break it down to simple logical “if-then” statements so the computer can analyze the next step to take in accordance with the opponent. In this assignment students will study basic computer science strategies of Boolean logic. They will use logic gates to create a circuit of possible combinations of numbers to produce a certain outcome. In the accompanying journal students will reflect on computer science, logic, and how they interact.

Objective

Students will look through the eyes of a computer scientist to create a computer circuit of possible combinations of numbers to produce a certain outcome. They will also reflect on computer science logic and its’ parallel with mathematical logic.

Materials

Blank Computer Science Assignment
Writing Utensil

Overview

The type of thinking used in computer science is based in Boolean logic which is deep rooted in mathematical logic. In fact you may have computer science majors within your classroom. Also the high and low voltage values on a computer circuit correspond directly to the 1 and 0, respectively, on a truth table and after passing through certain gates the resulting voltage changes accordingly. These gates are parallel to disjunction, conjunction, and negation. A short summary of these gates is given below.
The NOT gate corresponds to negation (See Figure 1). The bubble on the end is sometimes referred to as an inversion bubble. If a high voltage signal (1) goes through the line, a low voltage signal (0) will result on the other side and vice versa.

![Figure 1: The NOT gate and Corresponding Truth Table]

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

NOTE: adding an inversion bubble to any of the gates below results in the negation of the statement.

The AND Gate corresponds to conjunction (SEE Figure 2). If high voltage signals come into the gate simultaneously from all leads, a high voltage signal will result, anything else will result in a low voltage signal.

![Figure 2: The AND gate and Corresponding Truth Table]

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The OR gate corresponds to disjunction (SEE Figure 3). If a high voltage signal comes into the gate from any or all wires a high voltage signal will result, if all low voltage gates come into the a low voltage signal will result.

![Figure 3: The OR gate and Corresponding Truth Table]

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
The XOR gate (SEE figure 4) is very similar to the OR gate (shown above). It corresponds directly to the *exclusive or* logical connector. It will produce a high voltage signal only if the signals are mixed, if high voltage signals come into the gates from all lines, a low voltage signal will result. The same is true for all low voltage signals.

![XOR gate and Corresponding Truth Table](image)

**Figure 4:** The XOR gate and Corresponding Truth Table

Complex compound statements can be represented in terms of computer circuits. For example, consider the statement: \((p \land q) \lor (p \land r)\). (SEE figure 5)

![Compound statement using computer circuits](image)

**Figure 5:** The compound statement \((p \land q) \lor (p \land r)\) using computer circuits

If we think of this diagram in terms of electrical circuits, if high voltage signals \((1)\) pass through \(p\) and \(q\) simultaneously, a high voltage signal will also come out of the AND gate and into the OR gate. The same is true of a high voltage signal \((1)\) passes through \(p\) and \(q\). If a high voltage signal is produced by either AND gate or both AND gates and that signal passes through the OR gate, a high voltage signal will also be produced by the OR gate.

In this assignment students will be using analyzing all of the possible combinations of numbers to produce a sum of 10. Make sure students know that the plus signs (+) in these sums denotes an algebra
plus, not to be confused with a Boolean Algebra plus sign which would denote an OR, (In Boolean Algebra notation \( a + b = a \lor b \)). We are dealing with sets of numbers where their sum is 10, for example 4+6 = 10, so 4 \( \land \) 6 is a set of numbers that sum to 10, in Boolean Algebra notation: 4 \( \cdot \) 6.

If you are studying Boolean notation within your course you could have students convert each statement within their circuit to the Boolean algebra notation. The fact that we are dealing with sums of numbers but the Boolean algebra notation is multiplication because we are wanting sets of numbers, both numbers have to have a truth value of 1 in order to have a sum may trip students up at first but it is important that they think through it and make the distinction in order to truly understand the logical processes behind Boolean Algebra Notation.

Students will need to have knowledge of the gates above and the logic behind basic truth tables and mathematical logic to successfully complete the assignment below.
Computer Science Assignment

You are given the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 and you want to know how many combinations of these numbers will result in the number 10 when added together. There is no limit to how many numbers you use, but each digit can only occur once. Create a computer circuit that corresponds to these combinations and compound statement. For example, the numbers 1, 2, 3, 4, 5 adding up to 6 would be as follows:

NOTE: Remember in this particular scenario, each number can only occur once.
Computer Science Assignment

Computer Circuit of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 combining using addition to the number 10
A Case Brief of a Proof

Abstract

Along with mathematics and philosophy, the study of law also values logic. This assignment combines the two. Students will be given a false proof, and they must analyze it in an adapted form of a case brief. A case brief is an analysis common to law students. They must use their logical reasoning to break down the ‘case’ and analyze its weaknesses.

Overview

Students will use their mathematical reasoning and logic skills to analyze a false proof and reveal its weaknesses in the form of a case brief.

Materials

Case Brief and False Proof Assignment

Computer with Word Processor

Overview

A case brief is an analytical paper that law students use to evaluate law cases. It includes an analysis of the case and their opinion and reasoning on the verdict of the case. In this assignment students use the same format as a case brief (outlined below) to create an in-depth analysis of a common mathematical fallacy. It is a research style paper that requires students to rationalize every step taken in the proof and also where it went wrong. This assignment should be given to students familiar with proof and requires background knowledge of algebra. It is designed to enrich student’s skills in proof building and deciphering logical steps. There is a basic outline of the case brief to be given to each student and a few different proofs to choose from so not every student is working with the same proof.
Case Brief of a Proof

A comprehensive brief includes the following elements:

1. Title and Citation
2. Facts of the Case
3. Issues
4. Decisions (Holdings)
5. Reasoning (Rationale)
6. Analysis

1. Title and Citation

The title of your Court Brief will be your name vs. the conclusion of the proof. It also should include, the section of the class you are taking, the professor, and any other identifying material that your professor so chooses.

2. Facts of the Case

This section of the Case Brief will include a summary of the proof, the logical steps taken within the proof and the rationale behind each step. This includes any laws used to make inferences and the breakdown of the proof. It should be written in the following format.

- A one sentence description of the nature of the proof, to serve as an introduction.
- A statement about why this proof is being presented in the court, and the relevance in the mathematical world if this proof is proven valid. Think about the broader picture, what would it mean if this proof is valid graphically, asymptotically, financially, scientifically, and in the real world?
- A presentation of the complete proof.
- The rationale of step taken in the proof.

3. Issues

This section analyzes the proof. It is your job to find any weak spots within the proof and what is causing them. If there are questions about the validity of the proof, this is where they are
addressed. Make sure that any contradictory steps within the proof are explicitly stated, this includes the portion of the proof where the error occurred, and the mathematical law that is violated.

4. Decisions

This is where you present your conclusion on the case. This can be a simple one sentence statement on what you think about the validity of the proof.

5. Reasoning

In this section you will present the chain of reasoning that led to your conclusion. It should be written in numbered logical order pointing out exactly where in the proof problems arose or why you think each step is valid and the proof is sound.

6. Analysis

Here the students should evaluate the significance of the proof, its relationship to other proofs, its place in history and what it says for future proofs. If there was a problem with the proof this is where students look at the broader picture and analyze how to avoid this mistake in future proofs.

A CAUTIONARY NOTE

Do not brief the case until you have read it through at least once and understand each logical step taken within the proof. Look for unarticulated premises, logical fallacies, and mathematical errors. Does the result violate your sense of mathematical truth and rightness? How can you remedy this?
Proof #1

Proof that: $1 = -1$

$$1 = \sqrt{1}$$

$$\sqrt{1} = \sqrt{(-1)(-1)}$$

$$\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$$

$$\sqrt{-1}\sqrt{-1} = i \times i$$

$$i^2 = -1$$

Therefore,

$$1 = -1$$
Proof #2

**Proof That: 1 = 2**

\[a = b\]

\[ab = b^2\]

\[ab - a^2 = b^2 - a^2\]

\[a(b - a) = (b - a)(b + a)\]

\[\frac{a(b - a)}{b - a} = \frac{(b - a)(b + a)}{b - a}\]

\[a = b + a\]

Therefore

\[1 = 1 + 1\]

\[1 = 2\]
Proof #3

Proof That: \( \frac{16}{64} = \frac{1}{4} \)

\[
\begin{align*}
\frac{16}{64} &= \frac{16}{64} \\
\frac{16}{64} &= \frac{1}{4}
\end{align*}
\]

Therefore,

\[
\begin{align*}
\frac{16}{64} &= \frac{1}{4}
\end{align*}
\]
Proof That: 0 = 2

\[ \cos^2 x = 1 - \sin^2 x \]

*Taking the square root of both sides yields:*

\[ \cos x = (1 - \sin^2 x)^{\frac{1}{2}} \]

So,

\[ 1 + \cos x = 1 + (1 - \sin^2 x)^{\frac{1}{2}} \]

Replacing x with \( \pi \) yields:

\[ 1 + \cos \pi = 1 + (1 - \sin^2 \pi)^{\frac{1}{2}} \]

\[ 1 - 1 = 1 + (1 - 0)^{\frac{1}{2}} \]

Therefore,

\[ 0 = 2 \]
Sample Case Brief vs. Proof #1: *proof that* \(1 = -1\)

Spring 2013

I. Facts of the Case

This particular proof utilizes algebra and complex numbers to prove that \(1 = -1\). This case is being brought before the court because by the rules of common mathematics its conclusion is false. If \(1\) did indeed equal \(-1\), the entire fabric of mathematics would begin to unravel. If \(1 = -1\) then there would be no need for imaginary numbers because the square root of \(1\) is defined, therefore the square root of \(-1\) must be defined since they are equal. So either all numbers are real, or all numbers would be imaginary, there would be no separation, also what could we infer about \(-2\) and \(2\) and all other numbers and their negative opposites? Are they equal as well, and if all numbers are equal to one another, in the financial world what would a deficit be? Would a surplus and a deficit be equal entities? As you can see this proof presents some valid concerns if proven to be true. The proof is presented as follows as well as the mathematical law used in reasoning:

**Proof that:** \(1 = -1\)

1) \(1 = \sqrt{1}\) 
   The Power Property of Equality
2) \(\sqrt{1} = \sqrt{(-1)(-1)}\) 
   Fundamental Property of Inverses
3) \(\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}\) 
   Distributive Property of Exponents over Multiplication
4) \(\sqrt{-1}\sqrt{-1} = i \times i\) 
   The Substitution Property of Equality and the Definition of \(i\)
5) \(i^2 = -1\) 
   The Definition of \(i\)

Therefore

6) \(1 = -1\) 
   Conclusion of Proof
II. Issues

By just glancing at the rationale behind each step of the proof above all seems valid, but there is a weakness that is easily overlooked at step 3 of the proof. In this step the proof assumes that $\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$. It is operating under the Distributive property of Exponents over Multiplication. This law is valid for positive real numbers but negative and complex numbers introduce error. It is tempting to transfer all of the rules from real numbers onto complex numbers but this is not always valid; as it turns out the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ only holds when both $a$ and $b$ are positive.

III. Decisions

I believe this proof to be invalid.

IV. Reasoning

This proof is invalid because it violates the Property of Distribution over multiplication by involving the number -1. I sensed a problem because at every other step of the proof, if taken out of the proof is valid. No matter what numbers are substituted in for -1 in step 3 of the proof, there is a problem in evaluation and a false conclusion results. Through that chain of reasoning, one could prove that any $x = -x$ which violates the law of inverse. No number would have an inverse such that $x + (-x) = 0$.

V. Analysis

Though this proof is invalid, it brings to light the dangers of generalizing rules for all sets of numbers. Just because the property of exponential distribution over multiplication holds for the real numbers does not mean that it holds for complex number systems.
### Journal and Reflections

*The Road Before Us and the Road Behind: History and Research Applications.*

**The Laws of Logic**

As seen in the study of the history of logic, though Aristotle and Euclid worked in vastly different areas, their logic was similar. Write a professional letter to Euclid and Aristotle proposing a meeting and a merger between the two. Make sure you highlight their similarities and also areas that each can benefit from this meeting.

---

**Additional Ponderings: Reflect on this quote:**

“It would be a mistake to suppose that philosophical and mathematical logic are completely separate subjects. Actually, there is a unity between them...any sharp lines between to two would be arbitrary.”

_Haskell B. Curry_ The Foundation of Mathematical Logic
Journal and Reflections

The Road Before Us and the Road Behind: History and Research Applications.

The Laws of Logic

As seen in the study of the history of logic, though Aristotle and Euclid worked in vastly different areas, their logic was similar. Write a professional letter to Euclid and Aristotle proposing a meeting and a merger between the two. Make sure you highlight their similarities and also areas that each can benefit from this meeting.

Dear Sirs,

I have been studying your two logical systems and I have noticed some blatant similarities between your logical styles. Your two areas of study are very different in nature, however, there are some obvious similarities between the two. Euclid’s axiomatic method operate on the same logical systems where Aristotle’s syllogism and demonstrative logic reside. Mathematics and Philosophy are currently taught in two different schools of study in schools, but this is a disservice to both because at their core they go hand in hand. The two you working collaboratively could make immense progress in the merger between philosophy and mathematics and reshape the current education system. Please reply to my proposal ASAP.

Kind Regards,

_________________________________________________________
_________________________________________________________
_________________________________________________________
_________________________________________________________

Additional Ponderings: Reflect on this quote:

“It would be a mistake to suppose that philosophical and mathematical logic are completely separate subjects. Actually, there is a unity between them...any sharp lines between to two would be arbitrary.”

Haskell B. Curry The Foundation of Mathematical Logic

Mathematical logic and philosophical logic are very similar at their core. However they were studied in different schools in early education. This was the detriment of both. Their union has transformed mathematics and logic. Advances in one can only help advances in the other.
### Journal and Reflections

*The Road Before Us and the Road Behind: History and Research Applications.*

**Computer Science Logic**

Computers can be programmed to perform in infinite number of tasks even to think like a human. Consider how a computer is programmed to play tic-tac-toe. The programmer must break the game down into logical analyzable steps. For example consider the following board:

\[
\begin{array}{c|c|c}
O & & \\ \\
O & X & \\ \\
 & & X \\
\end{array}
\]

You are playing x’s explain the most logical next move and why. A computer must be programmed to recognize and capitalize on this move. Reflect on computer programming and how it uses logic to solve algorithms such as these.

**Feedback**

---

**Additional Ponderings: Reflect on this quote:**

“Reducto ad absurdum…is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.”

*G.H. Hardy*
Journal and Reflections

The Road Before Us and the Road Behind: History and Research Applications.

Computer Science Logic

Computers can be programmed to perform an infinite number of tasks even to think like a human. Consider how a computer is programmed to play tic-tac-toe. The programmer must break the game down into logical analyzable steps. For example, consider the following board:

You are playing x’s explain the most logical next move and why. A computer must be programmed to recognize and capitalize on this move. Reflect on computer programming and how it uses logic to solve algorithms such as these.

The most logical next step would be to place an x in the left-most corner of the board. This will not only stop the O’s from getting 3 in a row, but also ensure the X’s win. I will have two in a row across the diagonal that I could capitalize on to win, but if the O’s places their next move there I also have a 3 in a row opportunity across the bottom row of the board. There is no way for O’s to win. Computer programmers must break task down into simple analyzable statements just as the ones above. They have to look at all scenarios and create a logical next move.

Feedback

Additional Ponderings: Reflect on this quote:

“Reducto ad absurdum…is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.”

G.H. Hardy

This quote captures the beauty of mathematics. The mathematician has the power to decide what they are going to prove, they get to propose the premises, conclusions, and create a mathematical proof using logic similar to a chess board. Mathematicians get to bring whatever they want to the table for proof, mathematics is infinite after all.
Works Cited


**NOTE SOURCED STILL NEEDED:**
Education Text book
Geometry Text book
Discrete Math Text book
Aristotle Article