Quaternions and 3D Rotations

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Contents

Chapter 1

Quaternions: Definition and Basic Properties

Quaternions may be thought of as 4-tuples of real numbers: $q = (q_0, q_1, q_2, q_3)$. Equivalently, quaternions may also be written in the form

$$q = q_0 + q_1 i + q_2 j + q_3 k \tag{1.0.0.1}$$

where

$$\begin{aligned} i &= (0, 1, 0, 0) \\ j &= (0, 0, 1, 0) \\ k &= (0, 0, 0, 1) \end{aligned} \tag{1.0.0.2}$$

are themselves 4-tuples which satisfy the following relations

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.0.0.3}$$

In R^3, a vector is defined as $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are vectors of the form

$$\begin{aligned} \mathbf{i} &= (1, 0, 0) \\ \mathbf{j} &= (0, 1, 0) \\ \mathbf{k} &= (0, 0, 1). \end{aligned}$$

Any vector in R^3 can be mapped to R^4 by $(v_x, v_y, v_z) \mapsto (0, v_x, v_y, v_z)$ This mapping is often represented by $\mathbf{v} = 0 + \mathbf{v} = v$. By convention, the i, j, and k components of a quaternion are usually written as vectors. Thus, any quaternion q may be written as

$$q = q_0 + \mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \tag{1.0.0.4}$$

where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{1.0.0.5}$$

Here, q_0 is referred as the *scalar* part of the quaternion and $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is referred to as the *vector* part of the quaternion.

The quaternions form a non-commutative division ring. This means that quaternions have all the properties of a field except that quaternions are *not* commutative under multiplication. A discussion of the various properties of quaternions follows.

1.1 Equality

Two quaternions, $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ are considered equal if and only if their corresponding scalar components are equal:

$$\begin{aligned} p_0 &= q_0 \\ p_1 &= q_1 \\ p_2 &= q_2 \\ p_3 &= q_3. \end{aligned}$$

1.2 Addition

Two quaternions, $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, can be added together by combining their corresponding scalar and vector components:

$$\begin{aligned} p + q &= (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) + (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= (p_0 + q_0) + (p_1 + q_1)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_3 + q_3)\mathbf{k}. \end{aligned}$$

The following subsections discuss various properties of the quaternions under addition.

1.2.1 Closed Under Addition

The addition of two quaternions, $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, will result in a third quaternion, $r = r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}$:

$$\begin{aligned} p + q &= p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} + q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \\ &= (p_0 + q_0) + (p_1 + q_1)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_3 + q_3)\mathbf{k} \\ &= r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k} \\ &= r. \end{aligned}$$

1.2.2 Additive Identity

The additive identity for quaternions is the zero quaternion, $0 = 0 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$:

$$\begin{aligned} q + 0 &= q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} + 0 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\ &= (q_0 + 0) + (q_1 + 0)\mathbf{i} + (q_2 + 0)\mathbf{j} + (q_3 + 0)\mathbf{k} \\ &= q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \\ &= q. \end{aligned}$$

1.2.3 Additive Inverse

The additive inverse for a quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is given by $-q = -q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$:

$$\begin{aligned} q + (-q) &= (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) + (-q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}) \\ &= (q_0 - q_0) + (q_1 - q_1)\mathbf{i} + (q_2 - q_2)\mathbf{j} + (q_3 - q_3)\mathbf{k} \\ &= 0. \end{aligned}$$

1.2.4 Commutative Under Addition

Commutativity means that the order in which two quaternions are added together does not matter. Thus, for any two quaternions, $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$,

$$\begin{aligned} p + q &= (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) + (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= (p_0 + q_0) + (p_1 + q_1)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_3 + q_3)\mathbf{k} \\ &= (q_0 + p_0) + (q_1 + p_1)\mathbf{i} + (q_2 + p_2)\mathbf{j} + (q_3 + p_3)\mathbf{k} \\ &= (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) + (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) \\ &= q + p. \end{aligned}$$

1.2.5 Associative Under Addition

Associativity means given any three quaternions under addition, the order in which they are grouped together does not matter. For any three quaternions, $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$, $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, and $r = r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}$, observe that

$$\begin{aligned} (p + q) + r &= \big[(p_0 + q_0) + (p_1 + q_1)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_3 + q_3)\mathbf{k}\big] \\ &\quad + \big[r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}\big] \\ &= \big[(p_0 + q_0) + r_0\big] + \big[(p_1 + q_1) + r_1\big]\mathbf{i} + \big[(p_2 + q_2) + r_2\big]\mathbf{j} \\ &\quad + \big[(p_3 + q_3) + r_3\big]\mathbf{k}. \end{aligned}$$

Because the real numbers are associative under addition, the scalar componenets can be manipulated. Thus,

$$\begin{aligned} (p + q) + r &= \big[p_0 + (q_0 + r_0)\big] + \big[p_1 + (q_1 + r_1)\big]\mathbf{i} + \big[p_2 + (q_2 + r_2)\big]\mathbf{j} \\ &\quad + \big[p_3 + (q_3 + r_3)\big]\mathbf{k} \\ &= \big[p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\big] + \big[(q_0 + r_0) + (q_1 + r_1)\mathbf{i} + (q_2 + r_2)\mathbf{j} \\ &\quad + (q_3 + r_3)\mathbf{k}\big] \\ &= p + (q + r). \end{aligned}$$

To summarize, the associativity of quaternions under addition means that the order in which three or more quaternions are combined does not matte; i.e. $(p+q)+r = p+(q+r)$.

1.3 Multiplication

For quaternions, there are two types of multiplication to discuss: scalar multiplication and the muiltiplication of two quaternions. Scalar multiplication involves the multiplication of a quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ by any scalar $s \in \mathbb{R}$. The multiplication of two quaternions involves multiplying two quaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. Various properties of quaternions that deal with multiplication will also be discussed in this section.

1.3.1 Scalar Multiplication

Multiplication of a quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ by a scalar s is defined as

$$sq = sq_0 + s\mathbf{q} = sq_0 + sq_1\mathbf{i} + sq_2\mathbf{j} + sq_3\mathbf{k}.$$

1.3.2 Multiplication of Two Quaternions

The multiplication of two quaternions, $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$, is defined as

$$pq = (p_0 + \mathbf{p})(q_0 + \mathbf{q}) = p_0q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{pq}. \tag{1.3.2.1}$$

At this point is in unclear how to expand the term "**pq**." This will be revisted in a later section.

1.3.3 Associative Under Multiplication

Quaternions are associative under multiplication. This means that for any three quaternions, $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$, $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, and $r = r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}$, the order in which the quaternions are grouped together under multiplication does not matter:

$$\begin{aligned}(pq)r &= (p_0q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{pq})(r_0 + \mathbf{r}) \\ &= p_0q_0r_0 + p_0q_0\mathbf{r} + p_0r_0\mathbf{q} + p_0\mathbf{qr} + q_0r_0\mathbf{p} + q_0\mathbf{pr} + r_0\mathbf{pq} + \mathbf{pqr}.\end{aligned}$$

Factoring p_0 out of the first four terms and $\mathbf{p}$ out of the last four terms,

$$\begin{aligned}(pq)r &= p_0(q_0r_0 + q_0\mathbf{r} + r_0\mathbf{q} + \mathbf{qr}) + \mathbf{p}(q_0r_0 + q_0\mathbf{r} + r_0\mathbf{q} + \mathbf{qr}) \\ &= (p_0 + \mathbf{p})(q_0r_0 + q_0\mathbf{r} + r_0\mathbf{q} + \mathbf{qr}).\end{aligned}$$

By definition, $qr = q_0r_0 + q_0\mathbf{r} + r_0\mathbf{q} + \mathbf{qr}$. Thus,

$$(pq)r = p(qr) \tag{1.3.3.1}$$

and it has been shown that quaternions are associative under multiplication.

1.3.4 Implications of $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$

The relationships $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ must *always* hold; some important relationships that follow from these statements will now be derived.

Since $\mathbf{i}^2 = \mathbf{ii} = \mathbf{ijk} = -1$, this implies that

$$\mathbf{i} = \mathbf{jk}. \tag{1.3.4.1}$$

Similarly, since $\mathbf{k}^2 = \mathbf{kk} = \mathbf{ijk} = -1$, then

$$\mathbf{k} = \mathbf{ij}. \tag{1.3.4.2}$$

Using equation (1.3.4.2) and multiplying by $\mathbf{j}$ on the right,

$$\mathbf{k}(\mathbf{j}) = \mathbf{ij}(\mathbf{j})$$

$$\mathbf{kj} = \mathbf{i}(\mathbf{jj}) = \mathbf{i}(\mathbf{j}^2) = \mathbf{i}(-1) = -\mathbf{i}$$

so

$$\mathbf{i} = -\mathbf{kj}. \tag{1.3.4.3}$$

From equations (1.3.4.1) and (1.3.4.3), the following relationships can be found for $\mathbf{i}$:

$$\mathbf{i} = \mathbf{jk} = -\mathbf{kj}.$$

Using equation (1.3.4.1) and multiplying on the left by $\mathbf{j}$,

$$(\mathbf{j})\mathbf{i} = (\mathbf{j})\mathbf{jk}$$

$$\mathbf{ji} = (\mathbf{jj})\mathbf{k} = (\mathbf{j}^2)\mathbf{k} = (-1)\mathbf{k} = -\mathbf{k}$$

so

$$\mathbf{k} = -\mathbf{ji}. \tag{1.3.4.4}$$

From equation (1.3.4.2) and (1.3.4.4), the following relationships are found for $\mathbf{k}$:

$$\mathbf{k} = \mathbf{ij} = -\mathbf{ji}.$$

Using equation (1.3.4.4) and multiplying on the right by $\mathbf{i}$,

$$\mathbf{k}(\mathbf{i}) = -\mathbf{ji}(\mathbf{i})$$

$$\mathbf{ki} = -\mathbf{j}(\mathbf{ii}) = -\mathbf{j}(\mathbf{i}^2) = -\mathbf{j}(-1) = \mathbf{j}.$$

Thus,

$$\mathbf{j} = \mathbf{ki}. \tag{1.3.4.5}$$

Using equation (1.3.4.2) and multiplying on the left by $\mathbf{i}$,

$$(\mathbf{i})\mathbf{k} = (\mathbf{i})\mathbf{ij}$$

$$\mathbf{ik} = (\mathbf{ii})\mathbf{j} = (\mathbf{i}^2)\mathbf{j} = (-1)\mathbf{j} = -\mathbf{j}.$$

Hence,

$$\mathbf{j} = -\mathbf{ik}. \tag{1.3.4.6}$$

From equations (1.3.4.5) and (1.3.4.6), the following relationships are found for $\mathbf{j}$:

$$\mathbf{j} = \mathbf{ki} = -\mathbf{ik}.$$

To summarize, when multiplying two quaternions together, the relationships $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ must always hold. These relationships imply the following:

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k} \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i} \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j} \end{aligned} \tag{1.3.4.7}$$

The above relationships are helpful when multiplying two quaternions together.

1.3.5 Multiplication of Two Quaternions Revisited

Earlier the multiplication of two quaternions, $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$, was defined as

$$pq = (p_0 + \mathbf{p})(q_0 + \mathbf{q}) = p_0 q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{pq}.$$

The last term, $\mathbf{pq}$, is the product of two vectors. This term can now be expanded to become

$$\begin{aligned} \mathbf{pq} &= (p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= p_1\mathbf{i}(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) + p_2\mathbf{j}(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) + p_3\mathbf{k}(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= p_1q_1\mathbf{ii} + p_1q_2\mathbf{ij} + p_1q_3\mathbf{ik} + p_2q_1\mathbf{ji} + p_2q_2\mathbf{jj} + p_2q_3\mathbf{jk} \\ &\quad + p_3q_1\mathbf{ki} + p_3q_2\mathbf{kj} + p_3q_3\mathbf{kk}. \end{aligned}$$

The relationships developed in the previous subsection (1.3.4.7), and the fact that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, the above equation can be further simplified as

$$\begin{aligned} \mathbf{pq} &= p_1q_1(-1) + p_1q_2\mathbf{k} + p_1q_3(-\mathbf{j}) + p_2q_1(-\mathbf{k}) + p_2q_2(-1) + p_2q_3\mathbf{i} \\ &\quad + p_3q_1\mathbf{j} + p_3q_2(-\mathbf{i}) + p_3q_3(-1) \\ &= -(p_1q_1 + p_2q_2 + p_3q_3) \\ &\quad + [(p_2q_3 - p_3q_2)\mathbf{i} + (p_3q_1 - p_1q_3)\mathbf{j} + (p_1q_2 - p_2q_1)\mathbf{k}]. \end{aligned}$$

In the above equation, the term $(p_1q_1 + p_2q_2 + p_3q_3)$ is the familiar dot product of two vectors. The term $[(p_2q_3 - p_3q_2)\mathbf{i} + (p_3q_1 - p_1q_3)\mathbf{j} + (p_1q_2 - p_2q_1)\mathbf{k}]$ is the cross product of two vectors. Thus,

$$\mathbf{pq} = -(\mathbf{p} \cdot \mathbf{q}) + \mathbf{p} \times \mathbf{q}. \tag{1.3.5.1}$$

The above equation says that the product of two vectors in R^4 results in a scalar (the dot product) and another vector (the cross product).

Plugging $\mathbf{pq} = -(\mathbf{p}\cdot\mathbf{q}) + \mathbf{p}\times\mathbf{q}$ into equation (1.3.2.1), the multiplication of two quaternions, $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$, is

$$\begin{aligned}\mathbf{pq} &= p_0q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{pq}\\ &= p_0q_0 + p_0\mathbf{q} + q_0\mathbf{p} + [-(\mathbf{p}\cdot\mathbf{q}) + \mathbf{p}\times\mathbf{q}].\end{aligned}$$

The terms p_0q_0 and $-(\mathbf{p}\cdot\mathbf{q})$ are both scalars. The other three terms are all vectors. Grouping scalar terms together and vector terms together, the equation for the multiplication of two quaternions becomes

$$\mathbf{pq} = p_0q_0 - (\mathbf{p}\cdot\mathbf{q}) + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}. \tag{1.3.5.2}$$

1.3.6 Closed Under Multiplication

The multiplication of any two quaternions, $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$, will result in a third quaternion, $r = r_0 + \mathbf{r}$:

$$\begin{aligned}pq &= p_0q_0 - (\mathbf{p}\cdot\mathbf{q}) + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}\\ &= (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_2) + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\mathbf{i}\\ &\quad + (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)\mathbf{j} + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)\mathbf{k}\\ &= r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}\\ &= r.\end{aligned}$$

1.3.7 Multiplicative Identity

The product of a quaternion $q = q_0 + \mathbf{q}$ and the multiplicative idenitity should return the quaternion q. For quaternions, the multiplicative identity is the quaternion with a scalar part of 1 and zero vector part:

$$\begin{aligned}q(1) &= (q_0 + \mathbf{q})(1 + \mathbf{0})\\ &= q_0(1) - (\mathbf{q}\cdot\mathbf{0}) + q_0(\mathbf{0}) + (1)\mathbf{q} + \mathbf{q}\times\mathbf{0}\\ &= q_0 + \mathbf{q}\\ &= q.\end{aligned}$$

If the order of multiplication is reversed:

$$\begin{aligned}(1)q &= (1 + \mathbf{0})(q_0 + \mathbf{q})\\ &= (1)q_0 - (\mathbf{0}\cdot\mathbf{q}) + (1)\mathbf{q} + q_0(\mathbf{0}) + \mathbf{0}\times\mathbf{q}\\ &= q_0 + \mathbf{q}\\ &= q.\end{aligned}$$

Since $q(1) = (1)q = q$, 1 is the multplicative identity for R^4.

1.3.8 Conjugate

The complex conjugate of a quaternion $q = q_0 + \mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined as $q^* = q_0 - \mathbf{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. Note that multiplication of a quaternion and its complex conjugate always results in a scalar product and is commutative. Multiplying q by q^*,

$$\begin{aligned} qq^* &= (q_0 + \mathbf{q})(q_0 - \mathbf{q}) \\ &= q_0^2 - (\mathbf{q} \cdot -\mathbf{q}) + q_0\mathbf{q} - q_0\mathbf{q} + \mathbf{q} \times (-\mathbf{q}). \end{aligned}$$

The cross product of $\mathbf{q}$ and $-\mathbf{q}$ can be simplified by pulling the negative out: $\mathbf{q} \times -\mathbf{q} = -(\mathbf{q} \times \mathbf{q})$. Recalling that the cross product of any vector with itself is always equal to zero ($\mathbf{q} \times \mathbf{q} = 0$), the above equation now becomes

$$\begin{aligned} qq^* &= q_0^2 + \mathbf{q} \cdot \mathbf{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2. \end{aligned}$$

Switching the order of multiplication and multiplying q^* by q,

$$\begin{aligned} q^*q &= (q_0 - \mathbf{q})(q_0 + \mathbf{q}) \\ &= q_0^2 - (-\mathbf{q} \cdot \mathbf{q}) + q_0\mathbf{q} - q_0\mathbf{q} + (-\mathbf{q}) \times \mathbf{q}. \end{aligned}$$

Again, the negative can be pulled out of the cross product: $(-\mathbf{q}) \times \mathbf{q} = -(\mathbf{q} \times \mathbf{q}) = 0$. The above equation can now be simplified as

$$\begin{aligned} q^*q &= q_0^2 + \mathbf{q} \cdot \mathbf{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2. \end{aligned}$$

It has now been shown that the product of a quaternion and its complex conjugate is commutative and is equal to

$$qq^* = q^*q = q_0^2 + q_1^2 + q_2^2 + q_3^2. \tag{1.3.8.1}$$

1.3.9 Norm

The norm of a quaternion $q = q_0 + \mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined as

$$\begin{aligned} N(q) &= \sqrt{q^*q} \\ &= \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \\ &= \|q\|. \end{aligned} \tag{1.3.9.1}$$

1.3.10 Multiplicative Inverse

Let $q = q_0 + \mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ be any quaternion. The multiplicative inverse of q is given by the quaternion q^{-1} which will return the multplicative identity when it multiplies q; i.e. q^{-1} is the quaternion that satisfies the equations $qq^{-1} = 1$ and $q^{-1}q = 1$.

To find q^{-1}, multiply $qq^{-1} = 1$ on the right by q^*:

$$\begin{aligned} q^*(qq^{-1}) &= q^*(1) \\ (q^*q)(q^{-1}) &= q^* \\ \|q\|^2 q^{-1} &= q^* \\ q^{-1} &= \frac{q^*}{\|q\|^2}. \end{aligned}$$

Now, $q^{-1}q = 1$ must be multiplied by q^* on the left:

$$\begin{aligned} (q^{-1}q)q^* &= (1)q^* \\ (q^{-1})(q^*q) &= q^* \\ q^{-1}\|q\|^2 &= q^* \\ q^{-1} &= \frac{q^*}{\|q\|^2}. \end{aligned}$$

Thus, the multiplicative inverse for quaternions is given by

$$q^{-1} = \frac{q^*}{\|q\|^2}.$$

1.4 Multiplication is Distributive over Addition

Quaternion multiplication is distributive over addition; i.e. $p(q+r) = pq+pr$. For any three quaternions, $p = p_0 + \mathbf{p}$, $q = q_0 + \mathbf{q}$, and $r = r_0 + \mathbf{r}$,

$$\begin{aligned} p(q+r) &= (p_0 + \mathbf{p})\big[(q_0 + r_0) + (\mathbf{q} + \mathbf{r})\big] \\ &= p_0(q_0 + r_0) - \mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) + p_0(\mathbf{q} + \mathbf{r}) + (q_0 + r_0)\mathbf{p} + \mathbf{p} \times (\mathbf{q} + \mathbf{r}). \end{aligned} \quad (1.4.0.1)$$

Both the dot product and cross product are distributive over addition, so the terms $-\mathbf{p}\cdot(\mathbf{q}+\mathbf{r})$ and $\mathbf{p} \times (\mathbf{q} + \mathbf{r})$ become

$$\begin{aligned} -\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) &= -\mathbf{p} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{r} \\ \mathbf{p} \times (\mathbf{q} + \mathbf{r}) &= \mathbf{p} \times \mathbf{q} + \mathbf{p} \times \mathbf{r}. \end{aligned}$$

Equation (1.4.0.1) now becomes

$$\begin{aligned} p(q+r) &= p_0(q_0 + r_0) - \mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) + p_0(\mathbf{q} + \mathbf{r}) + (q_0 + r_0)\mathbf{p} + \mathbf{p} \times (\mathbf{q} + \mathbf{r}) \\ &= p_0q_0 + p_0r_0 - \mathbf{p} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{r} + p_0\mathbf{q} + p_0\mathbf{r} + q_0\mathbf{p} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} + \mathbf{p} \times \mathbf{r} \\ &= (p_0q_0 - \mathbf{p} \cdot \mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}) + (p_0r_0 - \mathbf{p} \cdot \mathbf{r} + p_0\mathbf{r} + r_0\mathbf{p} + \mathbf{p} \times \mathbf{r}) \\ &= pq + pr. \end{aligned}$$

Since $p(q+r) = pq + pr$, quaternion multiplication is distributive over addition.

1.5 Remarks on Quaternion Multiplication and Commutativity

When quaternion multiplication was introduced, it was stated that quaternions are not commutative under multiplication. In equation form, this means that

$$pq \neq qp.$$

A closer look at the equation for multiplication,

$$pq = p_0 q_0 - (\mathbf{p} \cdot \mathbf{q}) + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q},$$

reveals that term $\mathbf{p} \times \mathbf{q}$ leads quaternion multiplication to not be commutative. This is because the cross product of two vectors is itself not commutative.

If the term $\mathbf{p} \times \mathbf{q} = 0$, then, in this *special case*, quaternion multiplication will be commutative.

The following example shows a case where quaternion multiplication is not commutative.

Example 1.5.1.

Let $p = 1 - 3\mathbf{j} + 2\mathbf{k}$ and $q = 5 + 2\mathbf{i} - \mathbf{j}$. Here $p_0 = 1$, $q_0 = 5$, $\mathbf{p} = -3\mathbf{j} + 2\mathbf{k}$, and $\mathbf{q} = 2\mathbf{i} - \mathbf{j}$. First, $pq = p_0 q_0 - (\mathbf{p} \cdot \mathbf{q}) + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}$ will be computed. The dot and cross product are

$$\mathbf{p} \cdot \mathbf{q} = \big[(0)(2) + (-3)(-1) + (2)(0)\big] = 3$$

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= \big[(-3)(0) - (2)(-1)\big]\mathbf{i} + \big[(2)(2) - (0(0)\big]\mathbf{j} + \big[(0)(-1) - (-3)(2)\big]\mathbf{k} \\ &= 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} \end{aligned}$$

so

$$\begin{aligned} pq &= p_0 q_0 - (\mathbf{p} \cdot \mathbf{q}) + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q} \\ &= 1(5) - 3 + 1(2\mathbf{i} - \mathbf{j}) + 5(-3\mathbf{j} + 2\mathbf{k}) + 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} \\ &= 2 + 4\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}. \end{aligned}$$

To find qp, the dot and cross product are

$$\mathbf{q} \cdot \mathbf{p} = \big[(2)(0) + (-1)(-3) + (0)(2)\big] = 3$$

$$\begin{aligned} \mathbf{q} \times \mathbf{p} &= \big[(-1)(2) - (0)(-3)\big]\mathbf{i} + \big[(0(0) - (2)(2)\big]\mathbf{j} + \big[(2)(-3) - (-1)(0)\big]\mathbf{k} \\ &= -2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k} \end{aligned}$$

so

$$\begin{aligned} qp &= q_0p_0 - (\mathbf{q}\cdot\mathbf{p}) + q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{q}\times\mathbf{p} \\ &= 5(1) - 3 + 5(-3\mathbf{j} + 2\mathbf{k}) + 1(2\mathbf{i} - \mathbf{j}) - 2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k} \\ &= 2 - 20\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Thus, $pq = 2 + 4\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}$ and $qp = 2 - 20\mathbf{j} + 4\mathbf{k}$. As expected $pq \neq qp$.

1.6 Summary of Algebraic Properties

Various properties of quaternions have now been discussed. A summary of these properties is shown in Table 1.1. The addition and multiplication of two quaternions, how to find the conjugate of a quaternion, and how to computer the norm of a quatenrion have also been discussed; Table 1.2 lists these useful quatenrion operations.

Table 1.1: List of Quaternion Properties

Closed under addition and multiplication.
Commutative under addition: $p + q = q + p$
Associative under addition: $(p + q) + r = p + (q + r)$
Associative under multiplication: $(pq)r = p(qr)$
An identity exists for addition: $q + 0 = q$
Ad identity exists for multiplication: $q(1) = q$
Inverse exists for addition: $-q = -q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$
Inverse exists for multiplication: $q^{-1} = \frac{q^*}{\|q\|^2}$
Multiplication is distributive over addition: $p(q + r) = pq + pr$
Quaternions are **NOT** commutative under multiplication: $pq \neq qp$

Table 1.2: Summary of Quaternion Operations

Addition: $p + q = (p_0 + q_0) + (p_1 + q_1)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_3 + q_3)\mathbf{k}$
Multiplication: $pq = p_0q_0 - (\mathbf{p}\cdot\mathbf{q}) + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}$
Conjugate of $q = q_0 + \mathbf{q}$: $q^* = q_0 - \mathbf{q}$
Norm: $\|q\|^2 = \sqrt{q^*q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

The most important thing to remember when working with quaternions is that, in general, **multiplication is *not* commutative**:

$$pq \neq qp.$$

Because quaternion multiplication is not commutative, one must always be careful with the order in which two or more quaternions are multiplied together.

Chapter 2

Quaternions and Rotations

In his youth, Sir William Hamilton spent a lot of time developing complex number theory. During his work with complex numbers, he realized that by multiplying a complex number of the form $z = x + iy$ by a number of the form $u = \cos\theta + i\sin\theta$, he could rotate the vector z in R^2. Hamilton's work with complex numbers led him to believe that since a vector could be rotated by multiplying two numbers together in two-dimensions, there should also be a way to rotate vectors in three-dimensions by multiplying two numbers together.

For more than a decade of his life, Hamilton became obsessed with developing the theory of "triples." Triples were numbers of the form $w = w_0 + w_1 i + w_2 j$ where i and j are of the form $(0, 1, 0)$ and $(0, 0, 1)$. No matter how he tried, Hamilton could not find a way to multiply two triples together. The reason for this is simple: multiplication is not closed under R^3 – what Hamilton had been trying to do was *impossible* in R^3.

One day in 1843, while walking across the Royal Canal in Dublin, Hamilton had a flash of insight and realized that he could rotate a vector in R^4 by multiplying three numbers of the form $q = q_0 + q_1 i + q_2 j + q_3 k$ together where the relationships $i^2 = j^2 = k^2 = ijk = -1$ must be satisfied. Hamilton was so afraid of forgetting this equation, that he carved it into the Broom Bridge. While Hamilton's original carving has long since faded, a plaque now marks the spot where Hamilton famously discovered quaternions.

A discussion of how to rotate a vector in three-dimensions by multiplying three quaternions together follows.

2.1 Rotations in Three-Space

In this section, a way to rotate the vector $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_x\mathbf{k}$ through an angle 2θ about some vector $\mathbf{u}$ will be presented. The vector $\mathbf{u}$ must be a unit vector and both $\mathbf{v}$ and $\mathbf{u}$ must pass through the origin. The vector $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$ is the image of $\mathbf{v}$ *after* rotation.

Both $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^3 and must be mapped to R^4 by introducing a zero scalar part:

$$v = 0 + \mathbf{v}$$

$$w = 0 + \mathbf{w}.$$

By convention, the quaternions $v = 0 + \mathbf{v}$ and $w = 0 + \mathbf{w}$ are usually represented by the

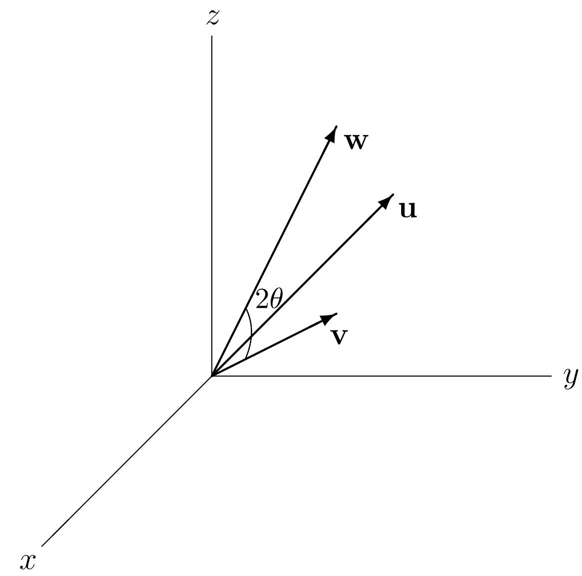

Figure 2.1: The vector $\mathbf{v}$ will be rotated clockwise about $\mathbf{u}$ through an angle of 2θ.

vectors $\mathbf{v}$ and $\mathbf{w}$. Both $\mathbf{v}$ and $\mathbf{w}$ will be used here.

Pre-multiplying $\mathbf{v}$ by a quaternion of the form $q = \cos\theta + \mathbf{u}\sin\theta$ and post-multiplying it by a quaternion of the form $q^* = \cos\theta - \mathbf{u}\sin\theta$, will result in a vector $\mathbf{w}$ that is the image of $\mathbf{v}$ rotated about $\mathbf{u}$ through an angle 2θ:

$$\begin{aligned} \mathbf{w} &= q\mathbf{v}q^* \\ &= (\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}(\cos\theta - \mathbf{u}\sin\theta). \end{aligned} \tag{2.1.1}$$

The following proof will show that equation (2.1.1) does indeed produce a vector $\mathbf{w}$ that is the image of $\mathbf{v}$ rotated about $\mathbf{u}$ through an angle 2θ.

2.2 Proof: $\mathbf{w} = q\mathbf{v}q^*$

It will now be shown that the equation $\mathbf{w} = q\mathbf{v}q^*$ results in a vector $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$ that is the image of $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ rotated about $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ through an angle 2θ. Here, $\mathbf{u}$ is a unit vector ($\|u\| = 1$) and both $\mathbf{u}$ and $\mathbf{v}$ pass through the origin. The quaternions q and q^* are have the form

$$\begin{aligned} q &= \cos\theta + \mathbf{u}\sin\theta \\ q^* &= \cos\theta - \mathbf{u}\sin\theta. \end{aligned}$$

The vector $\mathbf{v}$ will now be split into two components: $\mathbf{v}_\parallel$, the portion of $\mathbf{v}$ parallel to $\mathbf{u}$ and $\mathbf{v}_\perp$, the portion of $\mathbf{v}$ perpendicular to $\mathbf{u}$. Now,

$$\begin{aligned} \mathbf{w} &= q\mathbf{v}q^* \\ &= (\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}(\cos\theta - \mathbf{u}\sin\theta) \\ &= (\cos\theta + \mathbf{u}\sin\theta)(\mathbf{v}_\parallel + \mathbf{v}_\perp)(\cos\theta - \mathbf{u}\sin\theta). \end{aligned}$$

Quaternion multiplication is distributive over addition which allows the last equation to be split up into two terms:

$$\mathbf{w} = (\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_{\parallel}(\cos\theta - \mathbf{u}\sin\theta) + (\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_{\perp}(\cos\theta - \mathbf{u}\sin\theta). \tag{2.2.1}$$

It is easier to evaluate each term of the above equation individually.

Showing that $\mathbf{v}_{\parallel}$ is unchanged under rotation

Since $\mathbf{v}_{\parallel}$ is parallel to $\mathbf{u}$, it should be unchanged under rotation. It will now be shown that this is indeed the case. Expanding the term $(\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_{\parallel}(\cos\theta - \mathbf{u}\sin\theta)$,

$$(\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_{\parallel}(\cos\theta - \mathbf{u}\sin\theta) = \big[\cos\theta\mathbf{v}_{\parallel} + (\mathbf{u}\mathbf{v}_{\parallel})\sin\theta\big](\cos\theta - \mathbf{u}\sin\theta).$$

Note that the term $\mathbf{u}\mathbf{v}_{\parallel}$, is the product of two vectors. By equation (1.3.5.1), $\mathbf{u}\mathbf{v}_{\parallel} = \mathbf{u}\times\mathbf{v}_{\parallel} - \mathbf{u}\cdot\mathbf{v}_{\parallel}$. However, $\mathbf{u}$ and $\mathbf{v}_{\parallel}$ are parallel to one another so $\mathbf{u}\times\mathbf{v}_{\parallel} = 0$ and $\mathbf{u}\mathbf{v}_{\parallel} = -\mathbf{u}\cdot\mathbf{v}_{\parallel}$. Plugging this into the above equation,

$$\big[\cos\theta\mathbf{v}_{\parallel} + (\mathbf{u}\mathbf{v}_{\parallel})\sin\theta\big](\cos\theta - \mathbf{u}\sin\theta) = \big[\cos\theta\mathbf{v}_{\parallel} - (\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin\theta\big](\cos\theta - \mathbf{u}\sin\theta).$$

Expanding the right side,

$$\begin{aligned}\big[\cos\theta\mathbf{v}_{\parallel} - (\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin\theta\big](\cos\theta - \mathbf{u}\sin\theta) = {}& \cos^2\theta\mathbf{v}_{\parallel} - \mathbf{v}_{\parallel}\mathbf{u}\cos\theta\sin\theta - (\mathbf{u}\cdot\mathbf{v}_{\parallel})\cos\theta\sin\theta \\ & + \mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin^2\theta.\end{aligned}$$

The term $\mathbf{v}_{\parallel}\mathbf{u}$ can also be expanded using equation (1.3.5.1): $\mathbf{v}_{\parallel}\mathbf{u} = \mathbf{v}_{\parallel}\times\mathbf{u} - \mathbf{v}_{\parallel}\cdot\mathbf{u}$. Again, since $\mathbf{u}$ and $\mathbf{v}_{\parallel}$ are parallel to one another, $\mathbf{u}\times\mathbf{v}_{\parallel} = 0$ so $\mathbf{v}_{\parallel}\mathbf{u} = -\mathbf{v}_{\parallel}\cdot\mathbf{u}$. Plugging this into the above equation,

$$\begin{aligned}\cos^2\theta\mathbf{v}_{\parallel} & - \mathbf{v}_{\parallel}\mathbf{u}\cos\theta\sin\theta - (\mathbf{u}\cdot\mathbf{v}_{\parallel})\cos\theta\sin\theta + \mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin^2\theta \\ &= \cos^2\theta\mathbf{v}_{\parallel} + (\mathbf{v}_{\parallel}\cdot\mathbf{u})\cos\theta\sin\theta - (\mathbf{u}\cdot\mathbf{v}_{\parallel})\cos\theta\sin\theta + \mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin^2\theta \\ &= \cos^2\theta\mathbf{v}_{\parallel} + \big[(\mathbf{v}_{\parallel}\cdot\mathbf{u}) - (\mathbf{u}\cdot\mathbf{v}_{\parallel})\big]\cos\theta\sin\theta + \mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin^2\theta.\end{aligned}$$

The dot product of two vectors is commutative. This means that $(\mathbf{v}_{\parallel}\cdot\mathbf{u}) = (\mathbf{u}\cdot\mathbf{v}_{\parallel})$ so the term $(\mathbf{v}_{\parallel}\cdot\mathbf{u}) - (\mathbf{u}\cdot\mathbf{v}_{\parallel}) = (\mathbf{u}\cdot\mathbf{v}_{\parallel}) - (\mathbf{u}\cdot\mathbf{v}_{\parallel}) = 0$. The above equation now becomes

$$\cos^2\theta\mathbf{v}_{\parallel} + \big[(\mathbf{v}_{\parallel}\cdot\mathbf{u}) - (\mathbf{u}\cdot\mathbf{v}_{\parallel})\big]\cos\theta\sin\theta + \mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin^2\theta = \cos^2\theta\mathbf{v}_{\parallel} + \mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel})\sin^2\theta. \tag{2.2.2}$$

The product $\mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel})$ will now be simplified. By definition, the dot product of two vectors is defined as $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\phi$ where ϕ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{v}_{\parallel}$ was defined as being parallel to $\mathbf{u}$, ϕ is either $0°$ ($\mathbf{u}$ and $\mathbf{v}_{\parallel}$ are pointing in the same direction) or $180°$ ($\mathbf{u}$ and $\mathbf{v}_{\parallel}$ are pointing in opposite directions). If $\phi = 0°$,

$$\mathbf{u}(\mathbf{u}\cdot\mathbf{v}_{\parallel}) = \mathbf{u}(\|\mathbf{u}\|\|\mathbf{v}_{\parallel}\|\cos 0°).$$

Since $\cos 0^\circ = 1$ and $\|\mathbf{u}\| = 1$ (recall that we defined $\mathbf{u}$ to be a unit vector),

$$\mathbf{u}(\mathbf{u} \cdot \mathbf{v}_\parallel) = \|\mathbf{v}_\parallel\|\mathbf{u}.$$

Because the angle between $\mathbf{u}$ and $\mathbf{v}_\parallel$ is 0° and because $\mathbf{u}$ and $\mathbf{v}_\parallel$ both pass through the origin, $\mathbf{v}_\parallel = \|\mathbf{v}_\parallel\|\mathbf{u}$. Thus, $\mathbf{u}(\mathbf{u} \cdot \mathbf{v}_\parallel) = \|\mathbf{v}_\parallel\|\mathbf{u} = \mathbf{v}_\parallel$ when $\phi = 0^\circ$.

Now, if $\phi = 180^\circ$,

$$\mathbf{u}(\mathbf{u} \cdot \mathbf{v}_\parallel) = \mathbf{u}(\|\mathbf{u}\|\|\mathbf{v}_\parallel\| \cos 180^\circ).$$

Here $\cos 180^\circ = -1$ and $\|\mathbf{u}\| = 1$, so

$$\mathbf{u}(\mathbf{u} \cdot \mathbf{v}_\parallel) = -\|\mathbf{v}_\parallel\|\mathbf{u}.$$

Since the angle between $\mathbf{u}$ and $\mathbf{v}_\parallel$ is 180° and since $\mathbf{u}$ and $\mathbf{v}_\parallel$ both pass through the origin, $\mathbf{v}_\parallel = -\|\mathbf{v}_\parallel\|\mathbf{u}$. Thus, $\mathbf{u}(\mathbf{u} \cdot \mathbf{v}_\parallel) = -\|\mathbf{v}_\parallel\|\mathbf{u} = \mathbf{v}_\parallel$ when $\phi = 180^\circ$.

It has now been shown that $\mathbf{u}(\mathbf{u} \cdot \mathbf{v}_\parallel) = \mathbf{v}_\parallel$ whether the angle between $\mathbf{u}$ and $\mathbf{v}_\parallel$ is 0° or 180°. Plugging this into equation (2.2.2),

$$\begin{aligned}\cos^2\theta\mathbf{v}_\parallel + \mathbf{u}(\mathbf{u} \cdot \mathbf{v}_\parallel)\sin^2\theta &= \cos^2\theta\mathbf{v}_\parallel + \mathbf{v}_\parallel \sin^2\theta \\ &= (\cos^2\theta + \sin^2\theta)\mathbf{v}_\parallel.\end{aligned}$$

From basic trig, $\cos^2\theta + \sin^2\theta = 1$, so

$$(\cos^2\theta + \sin^2\theta)\mathbf{v}_\parallel = \mathbf{v}_\parallel.$$

It has now been shown that $(\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_\parallel(\cos\theta - \mathbf{u}\sin\theta) = \mathbf{v}_\parallel$, which is the first term in equation (2.2.1). As expected, $\mathbf{v}_\parallel$ is not changed when $\mathbf{v}$ is rotated about $\mathbf{u}$.

Showing that $\mathbf{v}_\perp$ is rotated through an angle 2θ upon rotation

Here, it will be shown that $\mathbf{v}_\perp$ will be rotated about $\mathbf{u}$ through an angle of 2θ. Expanding the second term in equation (2.2.2),

$$(\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_\perp(\cos\theta - \mathbf{u}\sin\theta) = (\cos\theta\mathbf{v}_\perp + \mathbf{u}\mathbf{v}_\perp\sin\theta)(\cos\theta - \mathbf{u}\sin\theta).$$

Here $\mathbf{u}\mathbf{v}_\perp$ is the product of two vectors, so it can be expanded using equation (1.3.5.1): $\mathbf{u}\mathbf{v}_\perp = \mathbf{u} \times \mathbf{v}_\perp - \mathbf{u} \cdot \mathbf{v}_\perp$. Since $\mathbf{u}$ is perpendicular to $\mathbf{v}_\perp$, the dot product of $\mathbf{u}$ and $\mathbf{v}_\perp$ is 0. This means that $\mathbf{u}\mathbf{v}_\perp$ simplifies as $\mathbf{u}\mathbf{v}_\perp = \mathbf{u} \times \mathbf{v}_\perp$. The above equation now becomes

$$(\cos\theta\mathbf{v}_\perp + \mathbf{u}\mathbf{v}_\perp\sin\theta)(\cos\theta - \mathbf{u}\sin\theta) = \big[\cos\theta\mathbf{v}_\perp + (\mathbf{u} \times \mathbf{v}_\perp)\sin\theta\big](\cos\theta - \mathbf{u}\sin\theta).$$

Expanding the right side,

$$\begin{aligned}\big[\cos\theta\mathbf{v}_\perp + (\mathbf{u} \times \mathbf{v}_\perp)\sin\theta\big](\cos\theta - \mathbf{u}\sin\theta) &= \cos^2\theta\mathbf{v}_\perp - \mathbf{v}_\perp\mathbf{u}\cos\theta\sin\theta + (\mathbf{u} \times \mathbf{v}_\perp)\cos\theta\sin\theta \\ &\quad -(\mathbf{u} \times \mathbf{v}_\perp)\mathbf{u}\sin^2\theta.\end{aligned}$$

The term $\mathbf{v}_\perp\mathbf{u}$ is the multiplication of two vectors. Using equation (1.3.5.1), $\mathbf{v}_\perp\mathbf{u} =$

$\mathbf{v}_\perp \times \mathbf{u} - \mathbf{v}_\perp \cdot \mathbf{u}$. Because $\mathbf{v}_\perp$ is perpendicular to $\mathbf{u}$, the dot product of $\mathbf{v}_\perp$ and $\mathbf{u}$ is 0. Thus, $\mathbf{v}_\perp\mathbf{u} = \mathbf{v}_\perp \times \mathbf{u}$ and the above equation becomes

$$\begin{aligned}\cos^2\theta\mathbf{v}_\perp &- \mathbf{v}_\perp\mathbf{u}\cos\theta\sin\theta + (\mathbf{u}\times\mathbf{v}_\perp)\cos\theta\sin\theta - (\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}\sin^2\theta \\ &= \cos^2\theta\mathbf{v}_\perp - (\mathbf{v}_\perp\times\mathbf{u})\cos\theta\sin\theta + (\mathbf{u}\times\mathbf{v}_\perp)\cos\theta\sin\theta - (\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}\sin^2\theta \\ &= \cos^2\theta\mathbf{v}_\perp + \big[-(\mathbf{v}_\perp\times\mathbf{u}) + (\mathbf{u}\times\mathbf{v}_\perp)\big]\cos\theta\sin\theta - (\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}\sin^2\theta.\end{aligned}$$

The cross product is anti-commutative. This means that $-(\mathbf{v}_\perp\times\mathbf{u}) = \mathbf{u}\times\mathbf{v}_\perp$. The term $\big[-(\mathbf{v}_\perp\times\mathbf{u})+(\mathbf{u}\times\mathbf{v}_\perp)\big]$ now becomes $(\mathbf{u}\times\mathbf{v}_\perp)+(\mathbf{u}\times\mathbf{v}_\perp) = 2(\mathbf{u}\times\mathbf{v}_\perp)$ so the above equation can now be written as

$$\begin{aligned}\cos^2\theta\mathbf{v}_\perp &+ \big[-(\mathbf{v}_\perp\times\mathbf{u}) + (\mathbf{u}\times\mathbf{v}_\perp)\big]\cos\theta\sin\theta - (\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}\sin^2\theta \\ &= \cos^2\theta\mathbf{v}_\perp + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta) - (\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}\sin^2\theta \\ &= \cos^2\theta\mathbf{v}_\perp - (\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}\sin^2\theta + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta). \qquad (2.2.3)\end{aligned}$$

The product $(\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}$ can be simplified in the above equation. First, note that $(\mathbf{u}\times\mathbf{v}_\perp)$ produces some vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}_\perp$. Since $(\mathbf{u}\times\mathbf{v}_\perp)$ is a vector, the product $(\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}$ can be expanded using equation (1.3.5.1): $(\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u} = (\mathbf{u}\times\mathbf{v}_\perp)\times\mathbf{u} - (\mathbf{u}\times\mathbf{v}_\perp)\cdot\mathbf{u}$. Because the vector produced by $(\mathbf{u}\times\mathbf{v}_\perp)$ is perpendicular to $\mathbf{u}$, then $(\mathbf{u}\times\mathbf{v}_\perp)\cdot\mathbf{u} = 0$. Thus, $(\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u} = (\mathbf{u}\times\mathbf{v}_\perp)\times\mathbf{u}$. Equation (2.2.3) becomes

$$\begin{aligned}\cos^2\theta\mathbf{v}_\perp &- (\mathbf{u}\times\mathbf{v}_\perp)\mathbf{u}\sin^2\theta + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta) \\ &= \cos^2\theta\mathbf{v}_\perp - \big[(\mathbf{u}\times\mathbf{v}_\perp)\times\mathbf{u}\big]\sin^2\theta + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta). \qquad (2.2.4)\end{aligned}$$

Using the identity $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + \mathbf{b}(\mathbf{a}\cdot\mathbf{c})$, the term $\big[(\mathbf{u}\times\mathbf{v}_\perp)\times\mathbf{u}\big]$ can be further expanded:

$$\big[(\mathbf{u}\times\mathbf{v}_\perp)\times\mathbf{u}\big] = -\mathbf{u}(\mathbf{v}_\perp\cdot\mathbf{u}) + \mathbf{v}_\perp(\mathbf{u}\cdot\mathbf{u}).$$

Since $\mathbf{v}_\perp$ is perpendicular to $\mathbf{u}$, $\mathbf{v}_\perp\cdot\mathbf{u} = 0$. Using the definition of the dot product, $\mathbf{u}\cdot\mathbf{u}$ becomes $\|\mathbf{u}\|^2\cos 0^\circ = 1^2 = 1$. Thus, $\big[(\mathbf{u}\times\mathbf{v}_\perp)\times\mathbf{u}\big] = \mathbf{v}_\perp$ so equation (2.2.4) becomes

$$\begin{aligned}\cos^2\theta\mathbf{v}_\perp &- \big[(\mathbf{u}\times\mathbf{v}_\perp)\times\mathbf{u}\big]\sin^2\theta + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta) \\ &= \cos^2\theta\mathbf{v}_\perp - \mathbf{v}_\perp\sin^2\theta + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta) \\ &= (\cos^2\theta - \sin^2\theta)\mathbf{v}_\perp + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta).\end{aligned}$$

Plugging in the identities $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $2\cos\theta\sin\theta = \sin 2\theta$, the above equation becomes

$$\begin{aligned}(\cos^2\theta - \sin^2\theta)\mathbf{v}_\perp + (\mathbf{u}\times\mathbf{v}_\perp)(2\cos\theta\sin\theta) &= \cos 2\theta\mathbf{v}_\perp + (\mathbf{u}\times\mathbf{v}_\perp)\sin 2\theta \\ &= \cos 2\theta\mathbf{v}_\perp + \sin 2\theta(\mathbf{u}\times\mathbf{v}_\perp). \qquad (2.2.5)\end{aligned}$$

It has just been shown that $(\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_\perp(\cos\theta - \mathbf{u}\sin\theta) = \cos 2\theta\mathbf{v}_\perp + \sin 2\theta(\mathbf{u}\times\mathbf{v}_\perp)$, which is the second term in equation (2.2.1). Earlier it was shown that $(\cos\theta +$

$\mathbf{u}\sin\theta)\mathbf{v}_{\|}(\cos\theta - \mathbf{u}\sin\theta) = \mathbf{v}_{\|}$. Thus,

$$\begin{aligned}
\mathbf{w} &= q\mathbf{v}q^* \\
&= (\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}(\cos\theta - \mathbf{u}\sin\theta) \\
&= (\cos\theta + \mathbf{u}\sin\theta)(\mathbf{v}_{\|} + \mathbf{v}_{\perp})(\cos\theta - \mathbf{u}\sin\theta) \\
&= (\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_{\|}(\cos\theta - \mathbf{u}\sin\theta) + (\cos\theta + \mathbf{u}\sin\theta)\mathbf{v}_{\perp}(\cos\theta - \mathbf{u}\sin\theta) \\
&= \mathbf{v}_{\|} + \cos 2\theta\mathbf{v}_{\perp} + \sin 2\theta(\mathbf{u}\times\mathbf{v}_{\perp}). \qquad (2.2.6)
\end{aligned}$$

Showing that $\|\mathbf{w}\| = \|\mathbf{v}\|$

$\mathbf{w}$ is supposed to be the image of the vector $\mathbf{v}$ after it has been rotated about $\mathbf{u}$ through an angle 2θ, so the length of $\mathbf{w}$ should be the same as the length of $\mathbf{v}$. Squaring the norm of equation (2.2.6),

$$\begin{aligned}
\|\mathbf{w}\|^2 &= \|q\mathbf{v}q^*\|^2 \\
&= \|\mathbf{v}_{\|} + \cos 2\theta\mathbf{v}_{\perp} + \sin 2\theta(\mathbf{u}\times\mathbf{v}_{\perp})\|^2.
\end{aligned}$$

Since $\mathbf{v}_{\|}$ is perpendicular to $\mathbf{v}_{\perp}$ and $\mathbf{u}\times\mathbf{v}_{\perp}$, equation (A.1) (see Appendix) can be used to split $\|\mathbf{v}_{\|} + \cos 2\theta\mathbf{v}_{\perp} + \sin 2\theta(\mathbf{u}\times\mathbf{v}_{\perp})\|^2$ into three terms:

$$\begin{aligned}
\|\mathbf{w}\|^2 &= \|\mathbf{v}_{\|} + \cos 2\theta\mathbf{v}_{\perp} + \sin 2\theta(\mathbf{u}\times\mathbf{v}_{\perp})\|^2 \\
&= \|\mathbf{v}_{\|}\|^2 + \|\cos 2\theta\mathbf{v}_{\perp}\|^2 + \|\sin 2\theta(\mathbf{u}\times\mathbf{v}_{\perp})\|^2 \\
&= \|\mathbf{v}_{\|}\|^2 + \|\mathbf{v}_{\perp}\|^2\cos^2 2\theta + \|(\mathbf{u}\times\mathbf{v}_{\perp})\|^2\sin^2 2\theta.
\end{aligned}$$

The magnitude of the cross product of $\mathbf{u}$ and $\mathbf{v}_{\perp}$ is given by $(\mathbf{u}\times\mathbf{v}_{\perp}) = \|\mathbf{u}\|\|\mathbf{v}_{\perp}\|\sin 90° = \|\mathbf{v}_{\perp}\|$ so

$$\begin{aligned}
\|\mathbf{w}\|^2 &= \|\mathbf{v}_{\|}\|^2 + \|\mathbf{v}_{\perp}\|^2\cos^2 2\theta + \|(\mathbf{u}\times\mathbf{v}_{\perp})\|^2\sin^2 2\theta \\
&= \|\mathbf{v}_{\|}\|^2 + \|\mathbf{v}_{\perp}\|^2\cos^2 2\theta + \|\|\mathbf{v}_{\perp}\|\|^2\sin^2 2\theta \\
&= \|\mathbf{v}_{\|}\|^2 + \|\mathbf{v}_{\perp}\|^2\cos^2 2\theta + \|\mathbf{v}_{\perp}\|^2\sin^2 2\theta \\
&= \|\mathbf{v}_{\|}\|^2 + \|\mathbf{v}_{\perp}\|^2(\cos^2 2\theta + \sin^2 2\theta).
\end{aligned}$$

Since $\cos^2 2\theta + \sin^2 2\theta = 1$,

$$\|\mathbf{w}\|^2 = \|\mathbf{v}_{\|}\|^2 + \|\mathbf{v}_{\perp}\|^2.$$

Using equation (A.1) again,

$$\begin{aligned}
\|\mathbf{w}\|^2 &= \|\mathbf{v}_{\|}\|^2 + \|\mathbf{v}_{\perp}\|^2 \\
&= \|\mathbf{v}_{\|} + \mathbf{v}_{\perp}\|^2 \\
&= \|\mathbf{v}\|^2.
\end{aligned}$$

Taking the squareroot of both sides of the above equation,

$$\|\mathbf{w}\| = \|\mathbf{v}\|.$$

Thus, is has been shown that the length of $\mathbf{w}$ is the same as the length of $\mathbf{v}$.

In summary, the product $q\mathbf{v}q^*$, where $q = \cos\theta + \mathbf{u}\sin\theta$ and $q^* = \cos\theta - \mathbf{u}\sin\theta$, produces a vector $\mathbf{w}$ that is the image of $\mathbf{v}$ rotated about some axis $\mathbf{u}$ through an angle of 2θ where both $\mathbf{v}$ and $\mathbf{u}$ pass through the origin.

2.3 Expanding $\mathbf{w} = q\mathbf{v}q^*$

Instead of multiplying $qvq^* = (cos\theta + \mathbf{u})\mathbf{v}(cos\theta - \mathbf{u})$ out every time, it is easiest to multiply it out once. After that, the vectors $\mathbf{v}$ and $\mathbf{u}$ and the angle θ can be plugged in and $\mathbf{w}$ can be easily computed. Expanding $\mathbf{w} = q\mathbf{v}q^*$,

$$\begin{aligned}\mathbf{w} &= q\mathbf{v}q^* \\ &= (cos\theta + \mathbf{u}\sin\theta)\mathbf{v}(cos\theta - \mathbf{u}\sin\theta) \\ &= (\mathbf{v}\cos\theta + \mathbf{uv}\sin\theta)(cos\theta - \mathbf{u}\sin\theta) \\ &= \mathbf{v}\cos^2\theta - \mathbf{vu}\cos\theta\sin\theta + \mathbf{uv}\cos\theta\sin\theta - \mathbf{uvu}\sin^2\theta \\ &= \mathbf{v}\cos^2\theta + \big[-\mathbf{vu} + \mathbf{uv}\big]\cos\theta\sin\theta - \mathbf{uvu}\sin^2\theta. \end{aligned} \tag{2.3.1}$$

Using equation (1.3.5.1) to expand the product of two vectors, the term $-\mathbf{vu} + \mathbf{uv}$ becomes

$$\begin{aligned}-\mathbf{vu} + \mathbf{uv} &= -\big[\mathbf{v}\times\mathbf{u} - \mathbf{v}\cdot\mathbf{u}\big] + \big[\mathbf{u}\times\mathbf{v} - \mathbf{u}\cdot\mathbf{v}\big] \\ &= -\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\times\mathbf{v} - \mathbf{v}\times\mathbf{u}.\end{aligned}$$

Since the dot product is commutative ($\mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v}$) and the cross product is anti-commutative ($\mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{v}$), the above equation becomes

$$\begin{aligned}-\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\times\mathbf{v} - \mathbf{v}\times\mathbf{u} &= -\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{v} \\ &= 2(\mathbf{u}\times\mathbf{v}).\end{aligned}$$

Plugging $-\mathbf{vu} + \mathbf{uv} = 2(\mathbf{u}\times\mathbf{v})$ into equation (2.3.1),

$$\begin{aligned}\mathbf{w} &= q\mathbf{v}q^* \\ &= \mathbf{v}\cos^2\theta + \big[-\mathbf{vu} + \mathbf{uv}\big]\cos\theta\sin\theta - \mathbf{uvu}\sin^2\theta \\ &= \mathbf{v}\cos^2\theta + 2(\mathbf{u}\times\mathbf{v})\cos\theta\sin\theta - \mathbf{uvu}\sin^2\theta.\end{aligned} \tag{2.3.2}$$

Using equation (1.3.5.1) to expand the product of $\mathbf{uvu}$,

$$\begin{aligned}\mathbf{uvu} &= (\mathbf{uv})\mathbf{u} \\ &= (\mathbf{u}\times\mathbf{v} - \mathbf{u}\cdot\mathbf{v})\mathbf{u} \\ &= (\mathbf{u}\times\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u} \\ &= (\mathbf{u}\times\mathbf{v})\times\mathbf{u} - (\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u}.\end{aligned}$$

Note that the vector $\mathbf{u}\times\mathbf{v}$ is perpendicular to $\mathbf{u}$, so the term $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = 0$. Using the identity $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + \mathbf{b}(\mathbf{a}\cdot\mathbf{c})$, $(\mathbf{u}\times\mathbf{v})\times\mathbf{u} = -\mathbf{u}(\mathbf{v}\cdot\mathbf{u}) + \mathbf{v}(\mathbf{u}\cdot\mathbf{u})$. The above equation can now be simplied as

$$\begin{aligned}\mathbf{uvu} &= (\mathbf{u} \times \mathbf{v}) \times \mathbf{u} - (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \\ &= -\mathbf{u}(\mathbf{v} \cdot \mathbf{u}) + \mathbf{v}(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \mathbf{v})\mathbf{u}.\end{aligned}$$

Recalling once again that the dot product is commutative ($\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}$) and since $\mathbf{u} \cdot \mathbf{u} = \|u\|^2 \cos 0^\circ = 1$,

$$\begin{aligned}\mathbf{uvu} &= -\mathbf{u}(\mathbf{v} \cdot \mathbf{u}) + \mathbf{v}(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \\ &= -\mathbf{u}(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v}(1) - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \\ &= -2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} + \mathbf{v}.\end{aligned}$$

Plugging $\mathbf{uvu} = -2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} + \mathbf{v}$ into equation (2.3.2),

$$\begin{aligned}\mathbf{w} &= q\mathbf{v}q^* \\ &= \mathbf{v}\cos^2\theta + 2(\mathbf{u} \times \mathbf{v})\cos\theta\sin\theta - \mathbf{uvu}\sin^2\theta \\ &= \mathbf{v}\cos^2\theta + 2(\mathbf{u} \times \mathbf{v})\cos\theta\sin\theta - (-2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} + \mathbf{v})\sin^2\theta \\ &= \mathbf{v}\cos^2\theta + 2(\mathbf{u} \times \mathbf{v})\cos\theta\sin\theta + 2(\mathbf{u} \cdot \mathbf{v})\sin^2\theta\mathbf{u} - \mathbf{v}\sin^2\theta \\ &= \mathbf{v}(\cos^2\theta - \sin^2\theta) + 2(\mathbf{u} \times \mathbf{v})\cos\theta\sin\theta + 2(\mathbf{u} \cdot \mathbf{v})\sin^2\theta\mathbf{u}.\end{aligned}$$

Using the identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$, the above equation can be simplied as

$$\mathbf{w} = q\mathbf{v}q^* = \mathbf{v}\cos 2\theta + (\mathbf{u} \times \mathbf{v})\sin 2\theta + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u}\sin^2\theta. \tag{2.3.3}$$

2.4 Rotation Algorithm

In order to rotate a vector $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ about an axis $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ through an angle 2θ where $\mathbf{v}$ and $\mathbf{u}$ pass through the origin, the following steps should be taken:

1. The vector $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ must be mapped from R^3 to R^4:

$$v = 0 + \mathbf{v}.$$

2. The axis of rotation must have length 1. If the desired axis of rotation, $\mathbf{m} = m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}$, does not have length 1, then $\mathbf{m}$ must be first normalized:

$$\mathbf{m} = \frac{m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}}{\sqrt{m_x^2 + m_y^2 + m_z^2}}.$$

3. Plug $\mathbf{v}$, $\mathbf{u}$, and 2θ into equation (2.3.3):

$$\mathbf{w} = \mathbf{v}\cos 2\theta + (\mathbf{u} \times \mathbf{v})\sin 2\theta + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u}\sin^2\theta.$$

4. The resulting quaternion $w = 0 + \mathbf{w}$ should have a zero scalar part. The vector part of w is the image of $\mathbf{v}$ after it has been rotated about $\mathbf{u}$ through an angle of 2θ.

2.5 Examples

The following examples show how to rotate a vector $\mathbf{v}$ about some axis $\mathbf{u}$ through an angle of 2θ. These examples make use of equation (2.3.3) to find the rotated vector $\mathbf{w}$. However, $\mathbf{w}$ could also be found directly by multiplying qvq* out using quaternion multiplication.

2.5.1 Example 1

Rotate the vector $\mathbf{v} = \mathbf{i}$ about the $\mathbf{j}$-axis through an angle of 120°. By equation (2.3.3), the rotated vector $\mathbf{w}$ will be

$$\mathbf{w} = \mathbf{v}\cos 2\theta + (\mathbf{u} \times \mathbf{v})\sin 2\theta + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u}\sin^2\theta$$

where

$$\begin{aligned}
\cos 2\theta &= \cos 120° = -\frac{1}{2} \\
\sin 2\theta &= \sin 120° = -\frac{\sqrt{3}}{2} \\
\sin^2\theta &= \sin^2 60° = \Big(\frac{\sqrt{3}}{2}\Big)^2 = \frac{3}{4}.
\end{aligned}$$

It is easiest to calculate the dot and cross product of $\mathbf{u} = \mathbf{j}$ and $\mathbf{v} = \mathbf{i}$ first and then plug their corresponding values into $\mathbf{w}$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \mathbf{j} \cdot \mathbf{i} \\
&= 0(1) + 1(0) + 0(0) \\
&= 0
\end{aligned}$$

and the cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \mathbf{j} \times \mathbf{i} \\
&= [1(0) - 0(0)]\mathbf{i} + [0(1) - 0(0)]\mathbf{j} + [0(0) - 1(1)]\mathbf{k} \\
&= -\mathbf{k}.
\end{aligned}$$

Plugging the corresponding dot and cross products into $\mathbf{w}$,

$$\begin{aligned}
\mathbf{w} &= \mathbf{v}\cos 2\theta + (\mathbf{u} \times \mathbf{v})\sin 2\theta + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u}\sin^2\theta \\
&= \mathbf{i}\Big(-\frac{1}{2}\Big) + \Big(-\mathbf{k}\Big)\Big(-\frac{\sqrt{3}}{2}\Big) + 2\Big(0\Big)\Big(\mathbf{j}\Big)\Big(\frac{3}{4}\Big)^2 \\
&= -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{k}.
\end{aligned}$$

As shown in Figure 2.2, the vector $\mathbf{v} = \mathbf{i}$ has been rotated 120° about the $\mathbf{j}$-axis into the vector $\mathbf{w} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{k}$.

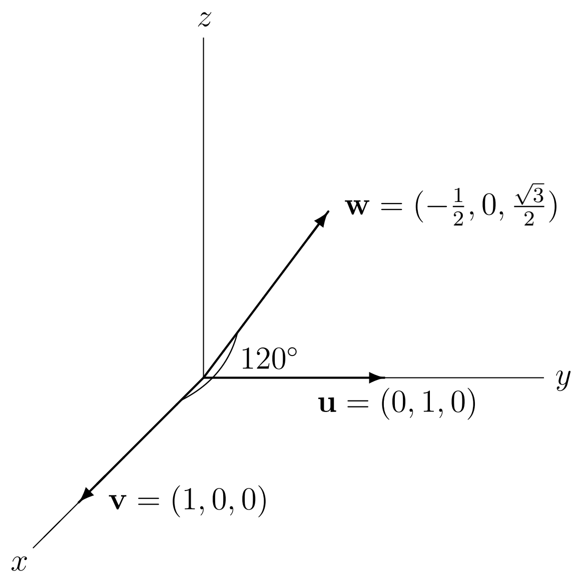

Figure 2.2: The vector $\mathbf{v} = \mathbf{i}$ is rotated 120° clockwise about $\mathbf{j}$ into $\mathbf{w} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{k}$.

2.5.2 Example 2

Rotate the vector $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ 90° about the axis $(\mathbf{i} + \mathbf{j})$. The first step is to normalize the axis of rotation:

$$\begin{aligned}\mathbf{u} &= \frac{1}{\sqrt{1^2 + 1^2 + 0^2}}\mathbf{i} + \frac{1}{\sqrt{1^2 + 1^2 + 0^2}}\mathbf{i} + \frac{0}{\sqrt{1^2 + 1^2 + 0^2}}\mathbf{k} \\ &= \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.\end{aligned}$$

Now, equation (2.3.3) can be used to find the rotated vector, $\mathbf{w}$,

$$\mathbf{w} = \mathbf{v}\cos 2\theta + (\mathbf{u} \times \mathbf{v})\sin 2\theta + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u}\sin^2\theta$$

where

$$\begin{aligned}\cos 2\theta &= \cos 90° = 0 \\ \sin 2\theta &= \sin 90° = 1 \\ \sin^2\theta &= \sin^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.\end{aligned}$$

Plugging the above trig identities in,

$$\begin{aligned}
\mathbf{w} &= \mathbf{v}\cos 90^\circ + (\mathbf{u}\times\mathbf{v})\sin 90^\circ + 2(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\sin^2 45^\circ \\
&= \mathbf{v}(0) + (\mathbf{u}\times\mathbf{v})(1) + 2(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\Big(\frac{1}{2}\Big) \\
&= (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})\mathbf{u}.
\end{aligned}$$

Again, it is easiest to calculate the dot and cross products of $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i}-5\mathbf{j}+2\mathbf{k}$ seperately and then plug them in to find $\mathbf{w}$. First, the dot product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \Big(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\Big)\cdot\Big(3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}\Big) \\
&= \frac{1}{\sqrt{2}}\Big(3\Big) + \frac{1}{\sqrt{2}}\Big(-5\Big) + 0\Big(2\Big) \\
&= \frac{3-5}{\sqrt{2}} \\
&= \frac{-2}{\sqrt{2}} \\
&= -\sqrt{2}.
\end{aligned}$$

The cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \Big(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\Big)\times\Big(3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}\Big) \\
&= \Big[\Big(\frac{1}{\sqrt{2}}\Big)\Big(2\Big) - 0\Big(5\Big)\Big]\mathbf{i} + \Big[0\Big(3\Big) - \frac{1}{\sqrt{2}}\Big(2\Big)\Big]\mathbf{j} + \Big[\frac{1}{\sqrt{2}}\Big(-5\Big) - \frac{1}{\sqrt{2}}\Big(3\Big)\Big]\mathbf{k} \\
&= \frac{2}{\sqrt{2}}\mathbf{i} - \frac{2}{\sqrt{2}}\mathbf{j} + \frac{-5-3}{\sqrt{2}}\mathbf{k} \\
&= \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} - 4\sqrt{2}\mathbf{k}.
\end{aligned}$$

The dot and cross product can now be plugged in to solve for $\mathbf{w}$,

$$\begin{aligned}
\mathbf{w} &= (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})\mathbf{u} \\
&= \Big(\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} - 4\sqrt{2}\mathbf{k}\Big) + \Big(-\sqrt{2}\Big)\Big(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\Big) \\
&= (\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} - 4\sqrt{2}\mathbf{k}) + (-\mathbf{i} - \mathbf{j}) \\
&= (-1+\sqrt{2})\mathbf{i} - (1+\sqrt{2})\mathbf{j} - 4\sqrt{2}\mathbf{k}.
\end{aligned}$$

As shown in Figure 2.3, the vector $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ has been rotated 90° about the axis $\mathbf{i}+\mathbf{j}$ into the vector $\mathbf{w} = (-1+\sqrt{2})\mathbf{i} - (1+\sqrt{2})\mathbf{j} - 4\sqrt{2}\mathbf{k}$.

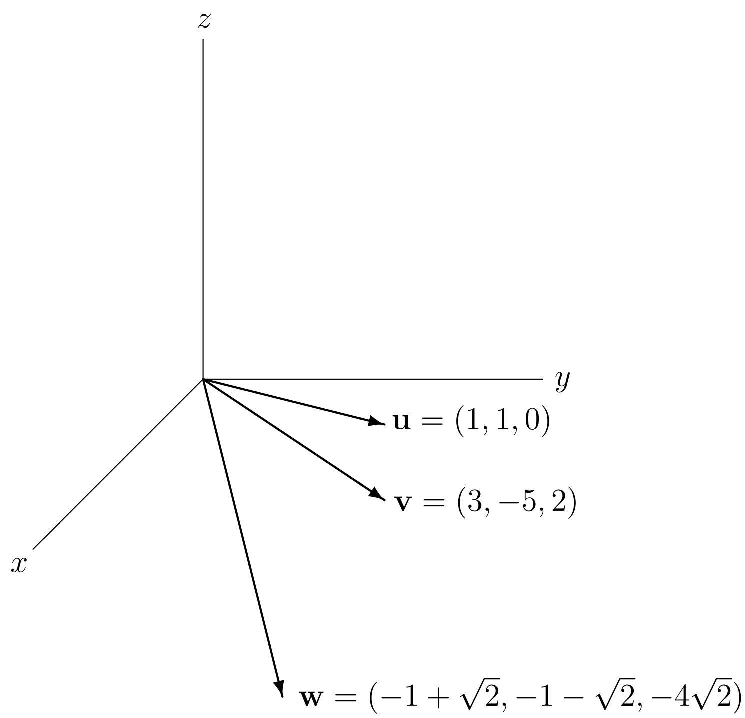

Figure 2.3: The vector $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ is rotated 90° clockwise about the axis $(\mathbf{i} + \mathbf{j})$ into $\mathbf{w} = (-1 + \sqrt{2})\mathbf{i} - (1 + \sqrt{2})\mathbf{j} - 4\sqrt{2}\mathbf{k}$.

Chapter 3

Rotating a Coordinate Frame S into Another Coordinate Frame S'

There are many real world situations where it is necessary to know how two coordinate frames, S and S', are related. Once the relationships between any two coordinate frames are known, any vector $\mathbf{v}$ in frame S can be rotated to its corresponding vector $\mathbf{v}'$ in S'.

In this chapter, a method for rotating a vector $\mathbf{v}$ in S into $\mathbf{v}'$ in S' will be presented. The method involves rotating a vector through two different axes and two different angles. Initially, this method will be presented without the use of quaternions. An algorithm that specially uses quaternions to carry out the two rotations will then be presented. Finally, an example using quaternions will be presented.

3.1 Generalized Two Axis-Angle Rotation Method

In order to use this method, the vectors $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{i} + v_z\mathbf{k}$ and $\mathbf{s} = s_x\mathbf{i} + s_y\mathbf{i} + s_z\mathbf{k}$ must be known and defined in the S frame. Their corresponding vectors, $\mathbf{v}' = v'_x\mathbf{i} + v'_y\mathbf{i} + v'_z\mathbf{k}$ and $\mathbf{s}' = s'_x\mathbf{i} + s'_y\mathbf{i} + s'_z\mathbf{k}$, in the S' frame must also be known. The vectors $\mathbf{v}'$ and $\mathbf{s}'$ must be defined with respect to the S frame.

1. Find the midpoint vector of $\mathbf{v}$ and $\mathbf{v}'$:

$$\mathbf{m} = \left(\frac{\mathbf{v_x} + \mathbf{v'_x}}{2}\right)\mathbf{i} + \left(\frac{\mathbf{v_y} + \mathbf{v'_y}}{2}\right)\mathbf{j} + \left(\frac{\mathbf{v_z} + \mathbf{v'_z}}{2}\right)\mathbf{k}.$$

2. Rotate $\mathbf{s}$ 180° about $\mathbf{m}$ into $\mathbf{s}^* = s^*_x\mathbf{i} + s^*_y\mathbf{j} + s^*_z\mathbf{k}$.

3. Find the angle of ϕ that will take $\mathbf{s}^*$ into $\mathbf{s}'$ when $\mathbf{s}$ is rotated about $\mathbf{v}'$.

4. Any vector $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$ can now be rotated into $\mathbf{r}' = r'_x\mathbf{i} + r'_y\mathbf{j} + r'_z\mathbf{k}$ by first rotating $\mathbf{r}$ 180° degrees about $\mathbf{m}$ followed by a rotation ϕ degrees about $\mathbf{v}'$.

3.2 Two Axis-Angle Rotation Method Using Quaternions

1. Define $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{i} + v_z\mathbf{k}$ and $\mathbf{s} = s_x\mathbf{i} + s_y\mathbf{i} + s_z\mathbf{k}$ to be any two vectors in the S frame. Let $\mathbf{v}' = v'_x\mathbf{i} + v'_y\mathbf{i} + v'_z\mathbf{k}$ and $\mathbf{s}' = s'_x\mathbf{i} + s'_y\mathbf{i} + s'_z\mathbf{k}$ be the corresponding vectors in the S' frame.

2. Find the midpoint vector of $\mathbf{v}$ and $\mathbf{v}'$:

$$\mathbf{m} = \Big(\frac{\mathbf{v_x} + \mathbf{v'_x}}{2}\Big)\mathbf{i} + \Big(\frac{\mathbf{v_y} + \mathbf{v'_y}}{2}\Big)\mathbf{j} + \Big(\frac{\mathbf{v_z} + \mathbf{v'_z}}{2}\Big)\mathbf{k}.$$

   Since $\mathbf{m}$ is the axis of the first rotation, it must be normalized:

$$\mathbf{m} = \frac{m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}}{\sqrt{m_x^2 + m_y^2 + m_z^2}}.$$

3. Equation (2.3.3) can be used to rotate $\mathbf{s}$ 180° about $\mathbf{m}$ into $\mathbf{s}^* = s^*_x\mathbf{i} + s^*_y\mathbf{j} + s^*_z\mathbf{k}$:

$$\begin{aligned}\mathbf{s}^* &= \mathbf{s}\cos 2\theta + (\mathbf{m}\times\mathbf{s})\sin 2\theta + 2(\mathbf{m}\cdot\mathbf{s})\mathbf{m}\sin^2\theta \\ &= \mathbf{s}\cos 180^\circ + (\mathbf{m}\times\mathbf{s})\sin 180^\circ + 2(\mathbf{m}\cdot\mathbf{s})\mathbf{m}\sin^2(90^\circ).\end{aligned}$$

   Plugging in $\cos 180^\circ = -1$, $\sin 180^\circ = 0$, and $\sin 90^\circ = 1$, the above equation becomes

$$\begin{aligned}\mathbf{s}^* &= \mathbf{s}(-1) + (\mathbf{m}\times\mathbf{s})(0) + 2(\mathbf{m}\cdot\mathbf{s})\mathbf{m}(1)^2 \\ &= -\mathbf{s} + 2(\mathbf{m}\cdot\mathbf{s})\mathbf{m}. \qquad (3.2.1)\end{aligned}$$

   When rotating $\mathbf{s}$ 180° about $\mathbf{m}$, it is easiest to use the equation above.

4. Before the second rotation about $\mathbf{v}'$ can be performed, $\mathbf{v}'$ must first be normalized:

$$\mathbf{v}' = \frac{v'_x\mathbf{i} + v'_y\mathbf{j} + v'_z\mathbf{k}}{\sqrt{{v'_x}^2 + {v'_y}^2 + {v'_z}^2}}.$$

5. Finding the angle ϕ that will take $\mathbf{s}$ into $\mathbf{s}'$ when $\mathbf{s}$ is rotated about $\mathbf{v}'$ can be quite difficult. The code provided in Appendix B uses a brute force method to estimate the angle.

   First, the code sets the angle to some minimum value and rotates the angle $\mathbf{s}^*$ about $\mathbf{v}'$ through that angle. It then compares each component of the rotated $\mathbf{s}^*$ and $\mathbf{s}'$ to see whether or not they are equal. If they are not equal, the angle is increasing by a set amount and the comparison is performed again. This test is repeated until some ϕ is found that rotates $\mathbf{s}^*$ into $\mathbf{s}'$ when rotated about $\mathbf{v}'$.

6. Any vector $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$ can now be rotated into $\mathbf{r}' = r'_x\mathbf{i} + r'_y\mathbf{j} + r'_z\mathbf{k}$ by first rotating $\mathbf{r}$ 180° degrees about $\mathbf{m}$ followed by a rotation ϕ degrees about $\mathbf{v}'$:

(a) $\mathbf{r}^* = -\mathbf{r} + 2(\mathbf{m} \cdot \mathbf{r})\mathbf{m}$

(b) $\mathbf{r}' = \mathbf{r}^* \cos\phi + (\mathbf{v}' \times \mathbf{r}^*) \sin\phi + 2(\mathbf{v}' \cdot \mathbf{r}^*)\mathbf{v}' \sin^2 \frac{\phi}{2}$

3.2.1 Example

Let $\mathbf{v} =< 1, 1, 1 >$ and $\mathbf{s} =< 0, 1, 0 >$ in the S frame. The corresponding vectors in the S' frame are $\mathbf{v}' =< -1, 1, 1 >$ and $\mathbf{s}' =< -1, 0, 0 >$.

The midpoint vector, $\mathbf{m}$ is found by adding the corresponding components of $\mathbf{v}$ and $\mathbf{v}'$ together and dividing by 2:

$$\begin{aligned} \mathbf{m} &= < \frac{1-1}{2}, \frac{1+1}{2}, \frac{1+1}{2} > \\ &= < 0, 1, 1 > . \end{aligned}$$

Since $\mathbf{m}$ and $\mathbf{v}'$ are the axes of rotation, both must be normalized:

$$\begin{aligned} \mathbf{m} &= < 0, \frac{1}{\sqrt{0^2+1^2+1^2}}, \frac{1}{\sqrt{0^2+1^2+1^2}} > \\ &= < 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \end{aligned}$$

$$\begin{aligned} \mathbf{v}' &= < \frac{-1}{\sqrt{(-1)^2+1^2+2^2}}, \frac{1}{\sqrt{(-1)^2+1^2+2^2}}, \frac{1}{\sqrt{(-1)^2+1^2+2^2}} > \\ &= < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > . \end{aligned}$$

Now, $\mathbf{s} =< 0, 1, 0 >$ can be rotated 180° about $\mathbf{m}$ using equation (3.2.1) to find that

$$\begin{aligned} \mathbf{s}^* &= -\mathbf{s} + 2(\mathbf{m} \cdot \mathbf{s})\mathbf{m} \\ &= - < 0, 1, 0 > +2\Big(< 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \cdot < 0, 0, 1 > \Big) < 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \\ &= - < 0, 1, 0 > +2\Big(0 * 0 + \frac{1}{\sqrt{2}} * 0 + \frac{1}{\sqrt{2}} * 1\Big) < 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \\ &= - < 0, 1, 0 > +2\Big(\frac{1}{\sqrt{2}}\Big) < 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \\ &= - < 0, 1, 0 > + < 0, 1, 1 > \\ &= < 0, 0, 1 > . \end{aligned}$$

At this point, the angle 2ϕ must be found that will rotate $\mathbf{s}^*$ into $\mathbf{s}'$ about the normalized $\mathbf{v}'$. Using the code provided in Appendix B, 2ϕ is found to be 240°. It will now be shown that 2ϕ does indeed rotate $\mathbf{s}^*$ into $\mathbf{s}'$:

$$\begin{aligned} \mathbf{s}' &= \mathbf{s}^* \cos\phi + (\mathbf{v}' \times \mathbf{s}^*) \sin\phi + 2(\mathbf{v}' \cdot \mathbf{s}^*)\mathbf{v}' \sin^2 \frac{\phi}{2} \\ &= \mathbf{s}^* \cos 240° + (\mathbf{v}' \times \mathbf{s}^*) \sin 240° + 2(\mathbf{v}' \cdot \mathbf{s}^*)\mathbf{v}' \sin^2 120° \end{aligned}$$

The cross product of $\mathbf{v}'$ and $\mathbf{s}^*$ is

$$\begin{aligned}\mathbf{v}' \times \mathbf{s}^* &= < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > \times < 0, 0, 1 > \\ &= < \Big(\frac{1}{\sqrt{3}}\Big)\Big(1\Big) - \Big(\frac{1}{\sqrt{3}}\Big)\Big(0\Big), \Big(\frac{1}{\sqrt{3}}\Big)\Big(0\Big) - \Big(-\frac{1}{\sqrt{3}}\Big)\Big(1\Big), \Big(-\frac{1}{\sqrt{3}}\Big)\Big(0\Big) - \Big(\frac{1}{\sqrt{3}}\Big)\Big(0\Big) > \\ &= < \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0 > .\end{aligned}$$

The dot product of $\mathbf{v}'$ and $\mathbf{s}^*$ is

$$\begin{aligned}\mathbf{v}' \cdot \mathbf{s}^* &= < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > \cdot < 0, 0, 1 > \\ &= \Big(-\frac{1}{\sqrt{3}}\Big)\Big(0\Big) + \Big(\frac{1}{\sqrt{3}}\Big)\Big(0\Big) + \Big(\frac{1}{\sqrt{3}}\Big)\Big(1\Big) \\ &= \frac{1}{\sqrt{3}}.\end{aligned}$$

Plugging in $\mathbf{v}' \times \mathbf{s}^* =< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0 >$, $\mathbf{v}' \cdot \mathbf{s}^* = \frac{1}{\sqrt{3}}$, $\cos 240° = -\frac{1}{2}$, $\sin 240° = -\frac{\sqrt{3}}{2}$, and $\sin^2 120° = -\Big(\frac{\sqrt{3}}{2}\Big)^2 = \frac{3}{4}$, the equation for $\mathbf{s}'$ becomes

$$\begin{aligned}\mathbf{s}' &= \mathbf{s}^* \cos 240° + (\mathbf{v}' \times \mathbf{s}^*) \sin 240° + 2(\mathbf{v}' \cdot \mathbf{s}^*)\mathbf{v}' \sin^2 120° \\ &= < 0, 0, 1 > \Big(-\frac{1}{2}\Big) + < \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0 > \Big(-\frac{\sqrt{3}}{2}\Big) + 2\Big(\frac{1}{\sqrt{3}}\Big) < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > \Big(\frac{3}{4}\Big) \\ &= < 0, 0, -\frac{1}{2} > + < -\frac{1}{2}, -\frac{1}{2}, 0 > + \Big(\frac{3}{2}\Big) < -\frac{1}{3}, \frac{1}{3}, \frac{1}{3} > \\ &= < -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} > + < -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} > \\ &= < -1, 0, 0 > .\end{aligned}$$

Rotating $\mathbf{s}^* =< 0, 0, 1 >$ 240° about $\mathbf{v}' =< -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$ does indeed rotate $\mathbf{s}^*$ into $\mathbf{s}' =< -1, 0, 0 >$.

Any vector $\mathbf{r}$ in the S frame can now be rotated into the S' frame by first rotating $\mathbf{r}$ 180° degrees about $\mathbf{m} =< 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$ into $\mathbf{r}^*$ and then rotating $\mathbf{r}^*$ 240° about $\mathbf{v}' =< -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$.

The vector $\mathbf{r} =< 2, 2, 0 >$ will now be rotated into the S' frame. First, $\mathbf{r}^*$ is found to be

$$\begin{aligned}\mathbf{r}^* &= -\mathbf{r} + 2(\mathbf{m} \cdot \mathbf{r})\mathbf{m} \\ &= - < 2, 2, 0 > + 2\Big(< 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \cdot < 2, 2, 0 > \Big) < 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \\ &= < -2, -2, 0 > + 2\Big(\frac{2}{\sqrt{2}}\Big) < 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > \\ &= < -2, -2, 0 > + < 0, 2, 2 > \\ &= < -2, 0, 2 > .\end{aligned}$$

The vector $\mathbf{r}^* =< -2, 0, 2 >$ must now be rotated 240° about $\mathbf{v}' =< -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$ to find $\mathbf{r}'$. Thus,

$$\begin{aligned}\mathbf{r}' &= \mathbf{r}^* \cos\phi + (\mathbf{v}' \times \mathbf{r}^*) \sin\phi + 2(\mathbf{v}' \cdot \mathbf{r}^*)\mathbf{v}' \sin^2 \frac{\phi}{2} \\ &= \mathbf{r}^* \cos 240^\circ + (\mathbf{v}' \times \mathbf{r}^*) \sin 240^\circ + 2(\mathbf{v}' \cdot \mathbf{s}^*)\mathbf{v}' \sin^2 120^\circ\end{aligned}$$

The cross product of $\mathbf{v}'$ and $\mathbf{r}^*$ is

$$\begin{aligned}\mathbf{v}' \times \mathbf{r}^* &= < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > \times < -2, 0, 2 > \\ &= < \Big(\frac{1}{\sqrt{3}}\Big)\Big(2\Big) - \Big(\frac{1}{\sqrt{3}}\Big)\Big(0\Big), \Big(\frac{1}{\sqrt{3}}\Big)\Big(-2\Big) - \Big(-\frac{1}{\sqrt{3}}\Big)\Big(2\Big), \Big(-\frac{1}{\sqrt{3}}\Big)\Big(0\Big) - \Big(\frac{1}{\sqrt{3}}\Big)\Big(-2\Big) > \\ &= < \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} > \\ &= < \frac{2}{\sqrt{3}}, 0, \frac{2}{\sqrt{3}} > .\end{aligned}$$

The dot product of $\mathbf{v}'$ and $\mathbf{r}^*$ is

$$\begin{aligned}\mathbf{v}' \cdot \mathbf{s}^* &= < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > \cdot < -2, 0, 2 > \\ &= \Big(-\frac{1}{\sqrt{3}}\Big)\Big(-2\Big) + \Big(\frac{1}{\sqrt{3}}\Big)\Big(0\Big) + \Big(\frac{1}{\sqrt{3}}\Big)\Big(2\Big) \\ &= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} \\ &= \frac{4}{\sqrt{3}}.\end{aligned}$$

Plugging $\mathbf{v}' \times \mathbf{r}^* =< \frac{2}{\sqrt{3}}, 0, \frac{2}{\sqrt{3}} >$, $\mathbf{v}' \cdot \mathbf{s}^* = \frac{4}{\sqrt{3}}$, $\cos 240^\circ = -\frac{1}{2}$, $\sin 240^\circ = -\frac{\sqrt{3}}{2}$, and $\sin^2 120^\circ = -\Big(\frac{\sqrt{3}}{2}\Big)^2 = \frac{3}{4}$, the equation for $\mathbf{r}'$ becomes

$$\begin{aligned}\mathbf{r}' &= \mathbf{r}^* \cos 240^\circ + (\mathbf{v}' \times \mathbf{r}^*) \sin 240^\circ + 2(\mathbf{v}' \cdot \mathbf{s}^*)\mathbf{v}' \sin^2 120^\circ \\ &= < -2, 0, 2 > \Big(-\frac{1}{2}\Big) + < \frac{2}{\sqrt{3}}, 0, \frac{2}{\sqrt{3}} > \Big(-\frac{\sqrt{3}}{2}\Big) + 2\Big(\frac{4}{\sqrt{3}}\Big) < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > \Big(\frac{3}{4}\Big) \\ &= < 1, 0, -1 > + < -1, 0, -1 > + < -2, 2, 2 > \\ &= < 0, 0, -2 > + < -2, 2, 2 > \\ &= < -2, 2, 0 > .\end{aligned}$$

The vector $\mathbf{r} =< 2, 2, 0 >$ in the S frame has now been rotated into $\mathbf{r}' =< -2, 2, 0 >$ in the S' frame.

Appendix A

Some Useful Equations

A.1 The Dot and Cross Product of Two Vectors

The dot and cross product of two vectors are two basic operations of linear algebra. The definition of each is given below.

A.1.1 The Dot Product

For any two vectors $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ and $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$, the dot product is defined as

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z. \tag{A.1}$$

Note that the dot product of two vectors results in a scalar. The dot product is also commutative; i.e. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

A.1.2 The Cross Product

For any two vectors $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ and $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$, the cross product is defined as

$$\mathbf{v} \times \mathbf{w} = (v_y w_z - v_z w_y)\mathbf{i} + (v_z w_x - v_x w_z)\mathbf{j} + (v_x w_y - v_y w_x)\mathbf{k}. \tag{A.1}$$

The cross product of two vectors always results in vector. The dot product of two vectors is anticommutative; i.e. $(\mathbf{v} \times \mathbf{w}) = -(\mathbf{w} \times \mathbf{v})$.

A.2 Useful Relationships

Some of the proofs in my thesis reference equations but do not bother to explain why the given equation is true. In these situations, how the equation was derived was not pertinent to the proof at hand. This section shows how some of these equations were derived.

A.2.1 Proof: If $\mathbf{v}$ is $\perp$ to $\mathbf{w}$, $\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$

Let $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ be a vector that is perpendicular to the vector $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$. By definition,

$$\begin{aligned} \|\mathbf{v}+\mathbf{w}\| &= \sqrt{(v_x+w_x)^2 + (v_y+w_y)^2 + (v_z+w_z)^2} \\ &= \sqrt{(v_x^2+w_x^2) + 2v_xw_x + (v_y^2+w_y^2) + 2v_yw_y + (v_z^2+w_z^2) + 2v_zw_z} \\ &= \sqrt{(v_x^2+w_x^2+v_y^2+w_y^2+v_z^2+w_z^2) + 2(v_xw_x+v_yw_y+v_zw_z)}. \end{aligned}$$

By definition, $\mathbf{v}\cdot\mathbf{w} = v_xw_x + v_yw_y + v_zw_z$. Since $\mathbf{v}$ is perpendicular to $\mathbf{w}$, $\mathbf{v}\cdot\mathbf{w} = 0$. The above equation now becomes

$$\begin{aligned} \|\mathbf{v}+\mathbf{w}\| &= \sqrt{(v_x^2+w_x^2+v_y^2+w_y^2+v_z^2+w_z^2) + 2(v_xw_x+v_yw_y+v_zw_z)} \\ &= \sqrt{(v_x^2+v_y^2+v_z^2) + (w_x^2+w_y^2+w_z^2)}. \end{aligned}$$

Note that $\|\mathbf{v}\|^2 = (v_x^2+v_y^2+v_z^2)$ and $\|\mathbf{w}\|^2 = (w_x^2+w_y^2+w_z^2)$ so

$$\begin{aligned} \|\mathbf{v}+\mathbf{w}\| &= \sqrt{(v_x^2+v_y^2+v_z^2) + (w_x^2+w_y^2+w_z^2)} \\ &= \sqrt{\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2}. \end{aligned}$$

Squaring both sides of the equation, it is found that

$$\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2. \tag{A.1}$$

Equation (A.1) hold for any vectors $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ and $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$ that are perpendicular to one another.

Appendix B

Source Code for the Two Axis-Angle Frame Rotation Method

Below the code is listed for the the axis-angle frame rotation method described in Chapter 3. This code was written using Microsoft Visual Studio 2008 Professional Edition. The WildMagic5.4 engine was also used to write code. WildMagic is available for free online at *http://www.geometrictools.com*.

The following program will rotate any vector **r** in a frame S into $\mathbf{r}'$ in the S' frame. The program requires the user to know the coordinates of two different vectors in both the S and S' frames. Once these coordiantes are known, the program will find an $\mathbf{r}'$ that corresponds to each r the user inputs.

B.1 C++ Code Listing

```
/*        TwoAxisAngleRotationMethod.cpp
 *
 *  Created by: Jeanette Schofield
 *                                  itzjeanette@yahoo.com
 *  Advisor: Dr. Wayne Keith, McMurry University
 *                                  keith.wayne@mcm.edu
 *  Date Created: 4/8/2011
 *
 *  Program Description:
 *       Once two different vectors, v and s, are known in both the S and S' frame, any vector r
 *  can be rotated from frame S to S'.
 *
 *  This method details a way to rotate between the S and S' frame using only two rotations.
 *
 *       Vectors in the S Frame:
 *               v              - any arbitary vector in the S frame
 *               vPRIME  - the vector that corresponds to v in the S' frame defined with respect to the S frame.
 *                                   For example, if v=(0,1,0) and the S frame is related to the S' frame by a 120 degree rotation about the
```

```
 *                                x-axis, then v'=(0,0,1).
 *               m                - midpoint vector of v and v' (m=(1/2)*(v+v')
 *               s                - a second arbitrary vector in the S frame
 *               sPRIME  - the vector that corresponds to s in the S' frame defined with respect to the S frame.
 *               r                - any vector in the S frame
 *               rPRIME  - the vector that corresponds to r in the S' frame defined with respect to the S frame.
 */

#include "stdafx.h"
#include <math.h>                /* Needed for various math function (sin, cos, pow,...). */
#include <stdio.h>               /* Used to print output to the screen and to get input from the user. */
#include <iomanip>               /* Used to format output to the screen. */
/* Files from WildMagic5.4 engine.  Allow for the use of Vector3 and Quaternion objects. */
#include "Wm5Matrix3.h"
#include "Wm5Quaternion.h"

/* Constants */
#define PI 3.14159265

/* Functions */
double degToRad(double degree);
double radToDeg(double radian);
double getAngle(Wm5::Vector3 <double> rotationAxis, Wm5::Vector3 <double> vecStart, Wm5::Vector3 <double> vecFinish);
void setVector(Wm5::Vector3 <double> * vec);
void setMidpointVector(Wm5::Vector3 <double> * midpointVec, Wm5::Vector3 <double> vector1, Wm5::Vector3 <double> vector2);
void normalizeVector(Wm5::Vector3 <double> * vector);
Wm5::Vector3 <double> rotate180aboutm(Wm5::Vector3 <double> rotAxis, Wm5::Vector3 <double> vec_s);
void printVector(Wm5::Vector3 <double> vector);

int _tmain(int argc, _TCHAR* argv[])
{
        /* Creates a char that will be used later for loop control. */
        char YorN;

        /* Creates all the vectors that will be used in the program.  Each vector is initialized to <0,0,0>.*/
        Wm5::Vector3 <double> vec_v(0,0,0), vec_vPRIME(0,0,0), vec_m(0,0,0);
        Wm5::Vector3 <double> vec_s(0,0,0), vec_sPRIME(0,0,0);
        Wm5::Vector3 <double> vec_r(0,0,0), vec_rPRIME(0,0,0);

        /* Sets the number of decimal places that will be output. */
        std::cout << std::fixed << std::setprecision(3);

        std::cout << "Enter the coordinates for vector v." << std::endl;
        setVector(& vec_v);
        std::cout << "Enter the coordinates for vector v'." << std::endl;
```

```
        setVector(& vec_vPRIME);
        std::cout << "Enter the coordinates for vector s." << std::endl;
        setVector(& vec_s);
        std::cout << "Enter the coordinates for vector s'." << std::endl;
        setVector(& vec_sPRIME);

        /* Create a midpoint vector. */
        setMidpointVector(& vec_m, vec_v, vec_vPRIME);

        /* Normalize m and v'. */
        normalizeVector(& vec_m);
        normalizeVector(& vec_vPRIME);

        /* Rotate s 180 degrees about m. */
        vec_s = rotate180aboutm(vec_m, vec_s);

        /* Calculate the anglePHI that will rotate s into s_PRIME when s is rotated about vPRIME. */
        double anglePHI = getAngle(vec_vPRIME, vec_s, vec_sPRIME);

        /* Create a quaternion with an axis of rotation of vPRIME and rotate through anglePHI. */
        Wm5::Quaternion<double> axis_vPRIME_anglePHI(vec_vPRIME, anglePHI);

        /*
         * Start main loop that allows a user to input any vector in the S frame and find out its
         * coordinates with respect to the S' frame.
         */
        do {
                /* Get r. */
                std::cout << "Enter the coordinates of any vector r in the S frame." << std::endl;
                setVector(& vec_r);

                /* Rotate r 180 degrees about m. */
                vec_r = rotate180aboutm(vec_m, vec_r);
                /* Rotate r PHI degrees about v'. */
                vec_rPRIME = axis_vPRIME_anglePHI.Rotate(vec_r);

                /* Prints the coordinates of r' to the screen. */
                std::cout << "r': ";
                printVector(vec_rPRIME);
                std::cout << std::endl;

                std::cout << std::endl << "Do you want to input another vector? (Y/N): ";
                std::cin >> YorN;
                std::cin.ignore(1000, '\n');
                std::cout << std::endl;

                /* If the user has entered a lowercase 'y' or 'n,' converts it to 'Y' or 'N'. */
                YorN = toupper(YorN);
```

```cpp

                /* This while loop checks to make the input from the user is of the correct form. */
                while( YorN != 'Y' && YorN != 'N')
                {
                        std::cout << "Please enter a Y or N: ";
                        std::cin >> YorN;
                        std::cin.ignore(1000, '\n');

                        YorN = toupper(YorN);
                }

        } while (YorN == 'Y');

        /* Clear output buffer (allows user to just press "Enter" once to exit program). */
        std::cout << std::endl << "Press Enter to exit program." << std::endl;
        std::cin.ignore(1000, '\n');
        getchar();

        return 0;
}

/* Converts degrees to radians. */
double degToRad(double degree)
{
        return degree*(PI/180);
}

/* Converts radians to degrees */
double radToDeg(double radian)
{
        return radian*(180/PI);
}

/* Figures out the angle necessary to rotate vecStart into vecFinish about rotationAxis. */
double getAngle(Wm5::Vector3<double> rotationAxis, Wm5::Vector3<double> vecStart, Wm5::Vector3<double> vecFinish)
{
        const double step = 0.0001;
        const double delta = 0.001;
        double angle = step;

        Wm5::Vector3<double> newVec(vecStart);

        while((abs(vecFinish.X() - newVec.X()) > delta) || (abs(vecFinish.Y() - newVec.Y()) > delta) || (abs(vecFinish.Z() - newVec.Z()) > delta))
        {
                Wm5::Quaternion<double> q_qConj(rotationAxis, angle);
                newVec = q_qConj.Rotate(vecStart);

                angle += step;
```

```
        }

        return angle;
}

/* Gets x, y, and z components from the user for a given vector. */
void setVector(Wm5::Vector3 <double> * vec)
{
        double x, y, z;

        /* Begin entering data. */
        std::cout << "x: ";
        std::cin >> x;
        std::cout << "y: ";
        std::cin >> y;
        std::cout << "z: ";
        std::cin >> z;
        std::cout << std::endl;

        vec->X() = x;
        vec->Y() = y;
        vec->Z() = z;
}

/* Sets midpointVec to be a vector that is halfway in between vector 1 and
   vector 2. */
void setMidpointVector(Wm5::Vector3 <double> * midpointVec, Wm5::Vector3 <
   double> vector1, Wm5::Vector3 <double> vector2)
{
        midpointVec->X() = ( vector1.X() + vector2.X() ) / 2;
        midpointVec->Y() = ( vector1.Y() + vector2.Y() ) / 2;
        midpointVec->Z() = ( vector1.Z() + vector2.Z() ) / 2;
}

/* Normalizes a vector by taking each component of a vector v=<v_x,v_y,v_z>
   and diviing it by the squareroot of
 * (v_x)^2+(v_y)^2+(v_z)^2.
 */
void normalizeVector(Wm5::Vector3 <double> * vector)
{
        double length = sqrt(pow(vector->X(), 2) + pow(vector->Y(), 2) + pow(
            vector->Z(), 2));

        vector->X() /= length;
        vector->Y() /= length;
        vector->Z() /= length;
}

/* This function returns vec_v after it has been rotated 180 degrees about
   rotAxis. */
Wm5::Vector3 <double> rotate180aboutm(Wm5::Vector3 <double> rotAxis, Wm5::
   Vector3 <double> vec_v)
{
        return -vec_v+(2*rotAxis.Dot(vec_v))*rotAxis;
```

```
}

/* Prints an vector out in the form "<v_x, v_y, v_z". */
void printVector(Wm5::Vector3<double> vector)
{
        std::cout << "<" << vector.X() << ", " << vector.Y() << ", " << vector
            .Z() << ">";
}
```