Abstract

The development of game theory has far surpassed parlor games and tricks, but solutions to such simple games can be maddening to a player unaware of the strategy. As these games are studied, they are sorted into classes, and a particularly interesting class is normal two-person, impartial, perfect information games. The 100 game is a member of this class, and coincidentally obtains a tricky solution, as do many other games of this class. Each of these games has the same strategy and is directly related to a game called Nim. This research will demonstrate the connections between the solutions of the 100 Game and Nim.

Acknowledgements

I would like to thank Dr. Cindy Martin for her incredible support and patience as I struggled to swim through the pond of mud called a thesis. Special appreciation is needed for Beth Young who was always there to reminded me that math is not all to life, though I still seriously doubt that is true. I would also like to thank my devoted husband, who always carried me through my hardest days and cheered me on as I danced upon my happiest.

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1 History and Theorems

1.1 Introduction

Game theory is a branch of mathematics created to study conflict and opportunity, which encompasses the world in which we live. It seems intuitive that if conflict is so often present, there should be someway to analyze patterns or make generalizations about such encounters. However extracting general concepts from specifics is not an easy task. When studying game theory, we must be able to see the essentials of a game and not be deceived by specific details. Anatol Rapoport best explained this idea by comparing a geometer obtaining circularity from circular objects to a game theorist finding "essential aspects of decisions in conflict situations" [7].

We can easily see that a manhole, a penny, and a tire are all circular, so we can make assumptions about these items based on our knowledge of circular objects. But can we easily look at different games and see their basic similarities? And if so, can we apply our knowledge of this similarity to help make optimal decisions in these situations? This paper is designed to help understand the similarities between two-person zero-sum perfect information games, specifically Nim and the 100 game.

1.2 A Brief History

The idea that situations can be objectively analyzed to determine optimal decisions is not a recently invented concept. In fact, writings of such topics date back to the Babylonian Talmud (0-500AD) on the rules of division of estate among widows [9]. The recommendations of the Talmud vary based on the size of the estate and respectively correspond to the nucleolus of an appropriately defined game. This variation in estate division has been recognized as foresight to the development of the modern theory of cooperative games.

There is a period of little progress in the study of game theory, but beginning in the early 1700's and continuing into modern day, there have been constant advances and contributions to this topic. The first set of a continual series of progressions in game theory resulted in numerous findings on solutions and strategy, but were found to be limited in their application.

In 1713, Francis Waldegrave delivered the first recognized minimax mixed strategy solution to a two person game [9]. However, Waldegrave could not seem to identify the same results with other games of chance. One hundred years later, the same fate was to fall upon Augustin Cournot's "Researches into the Mathematical Principles of the Theory of Wealth." This writing provided another limited result of a modern day theorem, but in this case, the result was a restricted version to the Nash Equilibrium. Rapidly following Cournot, Charles Darwin discussed equilibrium of the sex ratio in evolutionary biology. Darwin gave no physical proof of this ratio or that the value of the ratio was 1:1, but this was a big stride in building bridges between game theory and the natural sciences.

As the turn of the century passed by, Ernst Zermelo continued the path of game inspired analysis, as demonstrated in his major paper "Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels" [9]. This paper gives the first acknowledged theorem of game theory, which deals with possible endings in a game of chess. Emile Borel found the minimax strategy for two-person games with three or five strategies along with other notable results, and the following year, in 1928, John Von Neumann, proved this minimax theorem with finitely many pure strategies. His paper also introduced the extensive form of a game and discussed the outcome if mixed strategies are present. In 1944, John von Neumann continued his great work and, alongside Oskar Morgenstern, established the basis for game theory, particularly two person zero-sum game theory. This is considered the beginning of the study of game theory as we know it today.

Following this great stride in discovery, numerous reviews and additions to game theory exploded within a few years. John Nash provided the next major contribution in both non-cooperative game theory and bargaining theory over a series of four papers between 1950 and 1953 [9]. Nash proved what is known as the "Nash Equilibrium" and the "Nash Bargaining Solution" and founded axiomatic bargaining theory. A huge boom of progress in game theory, and a great outreach of related topics saturated the rest of the 1950's throughout the late 1970's. Game theorists made advances in nearly every branch of game theory, and new applications were being found around every corner. Infinitely long games, N-person games, game-based philosophy, stochastic games, and decision theory were only the beginning of the thriving times. Discoveries in game theory related to economics, biology, psychology, actuarial science, warfare, and law immediately followed. The rush of information came to a slow pace entering the 1990's, and a steady rate of findings established itself as a common practice.

The extreme growth of the applications of game theory began by studying the idea of conflict, which must include at least two people for the purpose of games. However, there is a type of game theory that involved no conflict, just competition, and this is called combinatorial game theory. The major contributions to combinatorial game theory begin with Charles Bouton in 1902 with his complete analysis of Nim [2]. Bouton proved that this game does in fact have strategy and this strategy holds true even as the game becomes finitely large. Bouton's work is not generally considered groundbreaking, because it seemed to have limited application at the time in which it was written.

R.P. Sprague initially discovered and proved a theorem which ultimately states that all impartial combinatorial games are equivalent to a game of Nim, and therefore have the same strategy [6]. Sprague did not publish any follow up work and his discovery was not well known. Four years later in 1939, P.M. Grundy independently discovered the same theorem. Each instance of this discovery not only proved relationships between Nim and impartial combinatorial games, but it also reveals the strategy to solve any game the meets such requirements. Each mathematician observed that a Nim game is the sum of smaller Nim games, called a disjunctive sum, and that the operation of the function governing this strategy is partially abelian. In 1945, Richard K. Guy rediscovered this theorem known at that time as the Grundy Theorem, while attempting to solve a different game. Guy was unaware of Sprague's work, and so any publication on impartial combinatorial games referenced only Grundy.

Guy went on to co-author the groundbreaking book "Winning Ways for Your Mathematical Plays" along with John Conway and Elwyn Berlekamp [6]. These three mathematicians worked together to tear down the barriers in combinatorial game theory. Conway and Berlekamp independently embraced Sprague and Grundy's disjunctive sum concept, as it applied to games, and each published works with great success. Conway wrote "On Numbers and

Games" in 1976, and developed a new number system with his findings. Berlekamp observed that the strategy for combinatorial games that were not necessarily impartial still related to the disjunctive sum concept, which gave way to a new area of research.

The value that "Winning Ways for Your Mathematical Plays" held in advancement of combinatorial games is priceless, but a tangible outcome of this piece is the categorical structure of the subject [6]. Now, we can analyze combinatorial games within the following categories:

- Algorithmic Games
- Complexity
- Hot Games
- Impartial Games
- All-Small Games

1.3 Characteristics of Impartial Games

Within the five categories of combinatorial games, for the purpose of this paper, we will focus on impartial games. Many concepts, ideas, and terms were mentioned in the previous section without definition or explanation, and now we will attempt to understand their meaning and the interdependence they share. Games can be categorized depending on many different factors and attributes of each application. A game that can be studied by the rules and findings of game theory must possess the following features [7].

1. There must be at least two players.

2. The game begins by one or more of the players making a choice among a number of specified alternatives.

3. After the first alternative is chosen and executed, a certain situation results. This situation determines what alternatives are available to the player making the current decision.

4. Choices made by all players may or may not become known to everyone.

5. There exists a termination rule if a game is played in a terms of successive choices.

6. Every play of a game[1] ends in a certain situation. Each of these situations determines a payoff to each bona fide player[2]. A player is one who (1) makes choices and (2) receives payoffs.

[1]Note the difference in the words "game" and "play." Game is defined as the totality of rules which define it, while play is common phrasing for a game played to point of termination.

[2]Chance can be a player in a game, but not a bona fide player. Chance may choose arrangements of alternatives, but she receives no payoffs.

In order to have some direction, we can follow the aforementioned features of game and see how altering each feature creates a new category by which we can classify different types of games [6].

- **Two-person vs. N-person**: A two-person game is a game consisting of two players that may or may not be bona fide. An n-person game is game consisting of finitely many players, some of whom may or may not be bona fide.

- **Partisan vs. Impartial**: An impartial game is a game in which the available alternatives in a turn depend only on the position and not on which player is moving. Essentially, this means that the only difference between player 1 and player 2, is that player 1 goes first. An example of an impartial game is tic-tac-toe. Chess is not an example of impartial game because player 1 can move only the white pieces, and one must know whose turn it is in order to analyze possible alternatives.

- **Sequential vs. Simultaneous**: A simultaneous game is a game in which all players move at the same time. A sequential game is a game which all players have some knowledge of previous moves by other players, and moves individually, but not necessarily alternatively. An example of a simultaneous game is rock-paper-scissors, and an example of a sequential game is checkers.

- **Perfect Information vs. Imperfect Information**: Perfect information applies to a player's complete knowledge of other players' moves and available options. Imperfect information does not have to be complete, but if a player has only some or no knowledge of other players decisions and opportunity, the game is considered to have imperfect information.

- **Infinitely long games**: An infinitely long game is a game in which there are infinite alternatives for each player, and a winner is not known until after the moves are completed. The focus of these games are not on the best strategy, but on the existence of a strategy.

- **Combinatorial Games**: A combinatorial game is a game in which the difficulty of finding an optimal strategy is dependent upon the number of possible moves for each player.

- **Zero-sum vs. Non-zero-sum**: A zero-sum game is a game in which one wins exactly the amount one's opponents lose.

- **Cooperative vs. Non-cooperative**: A cooperative game is a game in which players can make and receive obligatory commitments, and players are working together for a maximum payoff. Non-cooperative games are games in which players cannot make commitments to one another and payoff cannot increase for both players in the place of a commitment.

- **Strategy vs. Optimal play**: A strategy is a sequence of moves or decisions in which a player chooses to play a game for a desired outcome. Optimal play is a strategy in which a player is guaranteed a win.

- **Misére vs. Normal form**: A game in which the last player to move wins is played in normal form. A game in which the last player to move loses is played in misére form.

Our focus is on games which are impartial combinatorial games that are two person, sequential, perfect information, finite, zero-sum, and played in normal form. Now that we have determined the type of games to be analyzed, we will briefly discuss a few topics necessary for comprehension of the ideas to follow.

1.4 Binary Conversion

Binary Conversions There are a few notations and processes of which the reader should understand in order to easily follow the methods and strategies discussed in this paper. Decimal to binary conversions of numbers is a process of changing a number that is based on 10 units, decimal, to a number based on 2 units, binary. Decimal numbers are what humans use in everyday life, but binary numbers are essential to the strategy behind the games to follow. The process of conversion is simply a repetition of division by two with a remainder. We will now convert the 123 in to a binary number. Begin by writing the integer divided by 2 and then the quotient without remainder.Then write the remainder in the box to the right of the quotient.

$Integer \div 2 = Quotient$	$Remainder$
$123 \div 2 = 61$	1

Now, underneath what has been written, continue with the quotient divided by two to equal the new quotient with the remainder in the box to the right.

$Integer \div 2 = Quotient$	$Remainder$
$123 \div 2 = 61$	1
$61 \div 2 = 30$	1

Repeat this process, until the quotient is 0.

$Integer \div 2 = Quotient$	$Remainder$
$123 \div 2 = 61$	1
$61 \div 2 = 30$	1
$30 \div 2 = 15$	0
$15 \div 2 = 7$	1
$7 \div 2 = 3$	1
$3 \div 2 = 1$	1
$1 \div 2 = 0$	1

Then we begin with the bottommost number in the right-hand box and write the numbers working our way up. In this case we have

$$123_{10} = 1111011_2$$

For the numbers we will convert to binary in this paper, or for any relevant example one would like to work out, the following table of binary numbers should suffice.

Decimal	Binary	Decimal	Binary
0	0000	8	1000
1	0001	9	1001
2	0010	10	1010
3	0011	11	1011
4	0100	12	1100
5	0101	13	1101
6	0110	14	1110
7	0111	15	1111
		16	10000

2 Nim Game

2.1 The History of Nim

As functions can have parent functions, think of games as having the ability to have parent games. The 100 Game, which will be discussed later on is a child of the parlor game commonly called “Nim.” The origins of the game of Nim vary as different cultures claim its invention, but Nim closely resembles an ancient Chinese game called jianshizi or “picking stones”[8]. Charles Bouton later published a full analysis of Nim in 1902 and coined its current name, but he did not explain its selection [2]. The word “nimm” is a German word meaning “to take”, but Bouton did not specifically note any relationship between these two [6]. Interestingly, Nim was one of the first computerized games in 1939 at the World Exhibition in New York, as a robot called Nimrod that would play Nim against a human opponent and regularly win [3]. The study of this game began very early in Game theory history, but is extremely important to the advancement of combinatorial mathematics.

2.2 Bouton’s Definition and Solution

Charles Bouton described Nim in his paper, “Nim, A Game with a Complete Mathematical Theory,” (see [2]) as follows:

> **1. Description of the Game** The game is played by two players, A and B. Upon a table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no two piles shall be equal at the beginning. A play is made as follows:- The player selects one of the piles, and from it takes as many counters as he chooses; one, two, ..., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins.

Given three piles of counters, A, B, and C such that A=3, B=4, and C=5, a typical game of Nim could be played between two players, X and Y, as follows.

A	B	C	Move
3	4	5	Player X begins by taking 2 from A.
1	4	5	Player Y takes 3 from C.
1	4	2	Player X takes 1 from B.
1	3	2	Player Y takes all of A.
0	3	2	Player X takes 1 from B.
0	2	2	Player Y takes all of C.
0	2	0	Player X takes all of B.
0	0	0	Player X wins.

Bouton's analysis of Nim was able to prove that if a player can leave a certain set of numbers upon the table, and after that play without mistake, that the opponent cannot win. Bouton defined this set of numbers as a *safe combination* [2]. Bouton derived the group of safe combinations by the following method:

> **2. Its Theory** A *safe combination* is determined as follows: Write the number of counters in each pile in the binary scale of notation, and place these numbers in three horizontal lines so the units are in the same vertical columns. If then the sum of each column is 2 or 0(i.e. congruent to 0, mod. 2), the set of numbers forms a safe combination.

The sum modulo two attained by adding these binary numbers is called a **nim value**.

Example 2.1 The piles 2, 9, and 11 would be added like so

$$\begin{array}{l} 0010, \\ 1001, \\ \underline{1011}, \\ 0000 \end{array}$$

to form a safe combination. Bouton then used this idea of safe combinations or Nim values of 0 to give the following theorems.

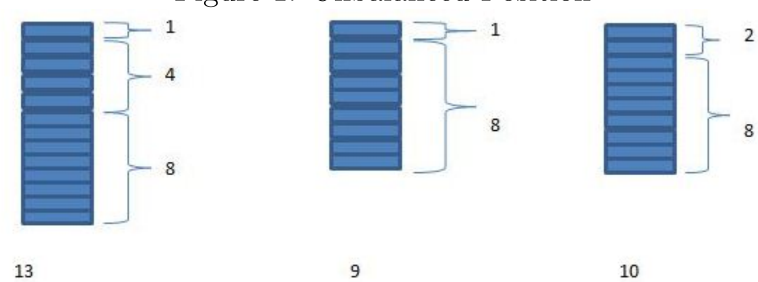
> **Theorem I** If A leaves a safe combination on the table, B cannot leave a safe combination on the table at his next move.

Theorem II If A leaves a safe combination on the table, and B diminishes one of the piles, A can always diminish one of the two remaining piles, and leave a safe combination.

Subpiles Although Bouton's methods do in fact allow the players to know whether or not the current state of the game is a safe combination, it does not provide a clear method to create a safe combination. If we consider the piles of counters as subpiles of powers of 2, we can modify Bouton's method into a more visually aesthetic and tangible method. By analyzing the number of subpiles of 2^n counters on the table, one can see how many counters to remove and from what pile. When a game is in a state of a safe combination, there will be even numbers of subpiles of 2^n counters on the table, and we call this situation a *balanced position* [1]. Furthermore, an uneven number of subpiles of 2^n is known as an *unbalanced position*. The goal of this strategy is to remove a certain number of counters from a single pile in order to leave a *safe combination*. This strategy is demonstrated through an example [1].

Example 2.2 We are given three piles, X, Y, and Z. Let pile X=13, Y=9, and Z=10. By dividing the piles into subpiles of powers of two, our game would resemble figure 1. Note that there are 3 subpiles of 8 on the table, automatically indicating the subpiles of 2^n are not even, denoting an unbalanced position.

Figure 1: Unbalanced Position



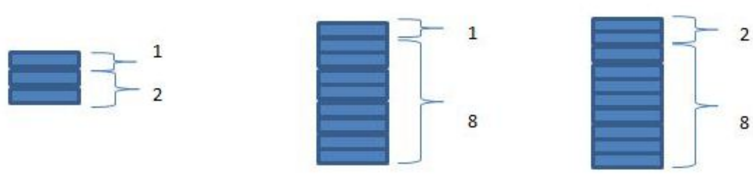
The next step in this strategy is to find out how many subpiles of different powers of two we have.
Table 2.3.1

Size	Number
8	3
4	1
2	1
1	2

We look to see which size subpiles are present in an uneven amount, and these subpiles are size 8, 4, and 2. The next step is to locate the largest subpile in an uneven amount, because this will be the pile from which we take to create a balanced position. In this example, the largest subpile in an uneven amount is subpile size 8. However, because there are 3 subpiles of 8, we must determine which pile to remove from. The optimal strategy calls to take from the largest subpile in the largest pile. This is subpile size 8 in pile X=13.

Since we will be taking away from X to balance Y and Z, we must look to see what size they contain. Y contains subpiles size 8,1, and Z contains subpiles size 8,2. Y and Z combined contain an even number of subpiles size 8, but only 1 subpile of size 1 and of size 2. In order to make this a balanced position, we must take away enough counters from X to leave two subpiles: one of size 1, and one of size 2. Therefore, we must remove 10 counters from X, leaving 3 total. Our table now resembles figure 2. Leaving the game in this state guarantees that if we repeat this process and play without error, we can undoubtedly attain a win.

Figure 2: Balanced Position



Disjunctive Sets Notice that a game of Nim can be broken down into separate games, each consisting of the single pile of counters [6]. Given a game of piles of 3, 4 and 5 counters, we can break this into games of single piles of 3, 4, 5 counters. The first player can take a whole pile on their first turn and win each game. However, when the games are added together, the strategy is not as simple, as previously demonstrated. For the full proof, see [5].

2.3 Safe Combinations Less than 16

The following are the 35 safe combinations of piles A, B, and C of whose piles are less than 16 and greater than 0. If there are three piles of counters on the table in these quantities, the game is balanced. Access to this table will be extremely helpful in the strategies of games to come.
Table 2.3.2 [2]

A	B	C	A	B	C	A	B	C
1	4	3	2	4	6	3	4	7
1	4	5	2	5	7	3	5	6
1	6	7	2	8	10	3	8	11
1	8	9	2	9	11	3	9	10
1	10	11	2	12	14	3	12	15
1	12	13	2	13	15	3	13	14
1	14	15						
5	8	13	6	8	14	7	8	15
5	9	12	6	9	15	7	9	14
5	10	15	6	10	12	7	10	13
5	11	14	6	11	13	7	11	12

3 The 100 Game

I stumbled upon the 100 Game at during a math seminar at the Texas Governor's School in 2008. The lecturer asked for students to volunteer as his opponent, and although numerous students tried with all their might, no one could defeat him. After inquiring about the 100 Game, I learned that it closely resemble the common children's game,"Race to Twenty," for which there is an easy solution [7]. In this demonstration, we will explore the solutions to the original 100 Game, and a three versions of the 100 Game with restrictions.

3.1 Case 1

Consider a game involving two players. Each player takes turns alternatively adding a number from 1 to 10 to the sum. The two players start from 0 and alternatively add a number from 1 to 10 to the sum. The first player to reach 100 wins.

To thoroughly explain game play, a few clarifications are in order.

Let the two players be X and Y, and let the sum be denoted Z. X begins by choosing a number to add to Z from the initial interval $R_0 = [1, 10]$. Note that this notation is to be interpreted as the numbers ranging from and including 1 to 10. After X's first turn $Z = X_0$. Y does likewise, and after Y adds to Z, then $Z = Y_0$.

The question behind the study of this game is whether or there there exists an optimal strategy.. Are there specific increments by which a player should increase Z that would guarantee a winning position? Yes, in fact there is a specific set of numbers called *safe numbers*. By increasing Z to these numbers at a given turn, a winning position is within reach. For this game in particular, the safe numbers are the set $W = 1, 12, 23, 34, 45, 56, 67, 78, 89$. We can derive this set of numbers by two methods. The first method is by working our way backwards from the safe number, W, to the first number within R_0. In this case $W = 100$ and $R_0 = [1, 10]$.

Since we do not know which player will win, for this simulation, let the two players be C and D. If C is to win, then we know his last move is $C_n = 100$ and his next to last move, C_{n-1}, would need to prevent D from reaching 100. This means that 99 should be the greatest number that D can guess. We know that D can add on to Z by any amount from 1 to 10, so if 99 is the greatest Z can be after D's last move, then 90 would be the lowest.

Therefore the range of D's last move would be $[90, 99]$. In order for C to force this range upon D, C_{n-2} must add enough to Z to reach 89. We can repeat this logic to see the following pattern:
Table 3.1.1

Player	Z
C_9	100
D_8	$[90, 99]$
C_8	89
D_7	$[79, 88]$
C_7	78
D_6	$[68, 77]$
C_6	67
D_5	$[57, 66]$
C_5	56
D_4	$[46, 55]$
C_4	45
D_3	$[35, 44]$
C_3	34
D_2	$[24, 33]$
C_2	23
D_1	$[13, 22]$
C_1	12
D_0	$[2, 11]$
C_0	1

The smallest safe number > 0 in this pattern is 1. Since $1 \in R_0 = [1, 10]$, then the player that goes first can always win. Note that although D is not given a specific number to guess, the pattern will still hold. Regardless of D's guess, C can still choose the next safe number in the series.

The second method is to take the order of the interval R_0, which in this case is 10 and add one, $10 + 1 = 11$. Then subtract multiple of $|R_0| + 1$ from 100, to get your safe numbers.

Table 3.1.2

Safe Numbers	$100 - n(\lvert R_0 \rvert + 1)$
89	$n = 1$
78	$n = 2$
67	$n = 3$
56	$n = 4$
45	$n = 5$
34	$n = 6$
23	$n = 7$
12	$n = 8$
1	$n = 9$

We can conclude that there is an optimal strategy in this case, but only for the player making the first move.

Notice that the first safe number is 1, because $100_{mod 11} = 1$. How would this strategy be affected if $R_0 = [1, 9]$ instead of $R_0 = [1, 9]$?

In this situation, most of the same rules apply as the original 100 Game, but a few modifications have been made.

Consider a game involving two players. Each player takes turns alternatively adding a number from 1 to 9 to the sum. The two players start from 0 and alternatively add a number from 1 to 9 to the sum. The first player to reach 100 wins.

Let the two players be X and Y, and let the sum be denoted Z. X begins by choosing a number to add to Z from the initial interval $R_0 = [1, 9]$. After X's first turn $Z = X_0$. Y does likewise, and after Y adds to Z, then $Z = Y_0$.

Once again, we do not know who will win out of X and Y, so we will use players C and D. Using the same logical process as earlier, we derive the following set of target numbers by the first method.

Table 3.1.3

Player	Z
C_9	100
D_9	$[91, 99]$
C_8	90
D_8	$[81, 89]$
C_7	80
D_7	$[71, 79]$
C_6	70
D_6	$[61, 69]$
C_5	60
D_5	$[51, 59]$
C_4	50
D_4	$[41, 49]$
C_3	40
D_3	$[31, 39]$
C_2	30
D_2	$[21, 29]$
C_1	20
D_1	$[11, 19]$
C_0	10
D_0	$[1, 9]$

The safe numbers found under these parameters are the set

W=[10,20,30,40,50,60,70,80,90]

Notice that the smallest safe number > 0 is 10 and $10 \notin R_0$. This means that the player who moves second will always have the winning position. We can conclude from these examples that if 100 modulo $|R_0| + 1 \notin R_0$, then the second player will win.

3.2 Case 2

We have seen that the optimal strategy for the previous cases is determined by the first player to select a safe number. What happens if the safe numbers are eliminated from game play, but not from the range, so players cannot choose these target numbers?

For this example, we have the same rules as the first case, but a player cannot change the sum to any of the numbers in the set

$$W_1 = \{1, 12, 23, 34, 45, 56, 67, 78, 89\}$$

This means that the first player's options are $R_0 = [1, 10] \backslash [1]$. Walking through the case backwards, let's suppose player C wins this game. He will have said 100 which means that player D would have had to said a number 90 and 99. In order for C to give D this range of choice, C must have said 89, but C cannot choose 89, because $89 \in T_1$. What is C to do? If he says 90, then D can say 100. Let's suppose C says 88; this would give D the options to choose $[89, 98]$, but since $89 \in T_1$, D's actual range of choice is $[89, 98] \backslash [89]$. This means that D can only say a number between 90 and 98, which ensures a win for C. But how is C to ensure an opportunity to say 88? The cycle repeats itself, and we repeat almost the same steps. In order for C to choose 88, he must give D a range of $[78, 87] \backslash [78]$, and in order to achieve this, C must choose, 77. Since $77 \notin T_1$ C can choose this number. We can repeat the latter process to find the following values.

Table 3.2.1

Player	Z
C	100
D	$[89, 98]$
C	88
D	$[78, 87]$
C	77
D	$[67, 76]$
C	66
D	$[56, 65]$
C	55
D	$[45, 54]$
C	44
D	$[34, 43]$
C	33
D	$[23, 32]$
C	22
D	$[12, 21]$
C	11
D	$[1, 10]$

Notice that the safe numbers in this game are multiples of 11 subtracted from 99, and each safe number shifted down one unit from the safe numbers in case 1. So, as the safe numbers in case 1 and case 2 are 11 units apart, we begin to see a pattern arise. How would strategy be affected if a player could not guess exactly 11 more than their previous turn?

3.3 Case 3

Missing Numbers Now, let's consider another slight variation of the 100 Game. Let the two players be X and Y, and let the sum be denoted Z. X begins by choosing a number to add to Z from the initial interval $R_0 = [1, 10]$. After X's first turn $Z = X_0$. Y does likewise, and after Y adds to Z, then $Z = Y_0$. However, in this variation, $X_n \neq X_{n-1} + 11$ $(n < 0)$. In other words, X cannot guess exactly 11 more than his previous guess. The first player to reach 100 wins the game.

We begin to recreate the tables from case 1 and 2 for the optimal strategies, but this case is slightly more difficult. We let C and D be the players in this game. Assuming C wins, we know that $C_n = 100$, and $D_n = [90, 99]$. In order to give D this range, C must choose 89, but if C chooses 89, C cannot choose 100. Therefore, C must choose 88 instead, leaving D with the range $[89, 98]$. Once again, if C chooses 88, there are two possibilities: D can choose a number between 90 and 98, and C can choose 100 ***or*** D can choose 89. Let's call this choice the detour route, because it has the same destination, but a different and longer route. If D chooses 89, then C will have a range of $[90, 99]$. Regardless of what C chooses in this case, D cannot choose 100, because D's previous choice was 89, which is 11 less than 100. Therefore, C can choose a number between $[90, 99] \backslash [99]$ since $88 + 11 = 99$, and then D is forced to say 99 since he can't say 100. Finally C is left with the opportunity to choose 100 and win the game. We can conclude that as long as C can choose 88, then a win is imminent. Now we must derive a strategy to allow C to choose 88. By following the same logic as just stated, we can create the following table.

Table 3.3.1

Player	Z
C	100
D	$[89, 98]$
C	88
D	$[77, 86]$
C	76
D	$[65, 74]$
C	64
D	$[53, 62]$
C	52
D	$[41, 50]$
C	40
D	$[29, 38]$
C	28
D	$[17, 26]$
C	16
D	$[5, 14]$
C	4

Notice that there are numbers that are not present between every range of D's guesses, and C's safe numbers. These numbers are 15,27,39,51,63,75,87, and 99. If on any turn projected in the table D chooses the smallest number available to him, we experience the detour route as previously mentioned. Also notice that because $4 \in R_0 = [1, 10]$, C is the first player, therefore the first player always has the opportunity to win.

3.4 Grundy-Values and Nim Values

Now that the strategies of the 100 Game in three cases have been explained, how can we connect this to the binary-focused strategy of Nim? An important theorem to the development of the connections between Nim and the 100 Game is the Sprague-Grundy Theorem.

Sprague-Grundy Theorem In combinatorial game theory, every impartial game under the normal play convention whose rules guarantee termination is equivalent Nim.

For the purposes of this theorem, it is important to understand the concept of a *cul-de-sac*. A cul-de-sac is a game in which every move results in a position closer to game termination; in other words, a player cannot move backwards. If a game is a cul-de-sac, it's positions can be divided into sets according to the greatest number of moves remaining to terminate the game [1].

We let $\{P_0\}$ be the set in which terminates the game and $\{P_1\}$ be the set in which next move will terminate the game. The set $\{P_n\}$ is the set with at most n moves required to terminate the game, and $\{P_n\}$ has a follower in $\{P_{n-1}\}$, and may have more within in other sets[1].

For example, in the original 100 game, the set $\{P_0\} = [100]$,because that is the only position which terminates the game. The set $\{P_1\} = \{90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$, because a player choosing any of these numbers will allow the next player to choose $\{P_0\}$=100, which terminates the game.

Now, we introduce the *Grundy-value*[1], a value assigned to each position in a set $\{P_n\}$.

- To each position in the set $\{P_0\}$, we assign the value 0.
- To each position in the set $\{P_1\}$, we assign the value 1.
- The set $\{P_2\}$ is assigned a value that is the lowest value not assigned to any of its immediate followers. The followers of a position in $\{P_2\}$ may be in $\{P_1\}$ and $\{P_0\}$, and therefore its followers may have a Grundy-value of 1 or 0, so in this case, $\{P_2\}$ has a value of 2. If all of $\{P_2\}$'s followers are in $\{P_1\}$, then all of $\{P_2\}$'s followers have a Grundy-value of 1. Hence, in this case the smallest value that is not assigned to a follower of $\{P_2\}$ is 0, so $\{P_2\}$ is assigned a value of 0.
- This process continues for all sets $\{P_n\}$ of positions in a game.

If we begin to analyze the positions of the 100 Game played , we attain these results:

- $[100] \in \{P_0\} = 0$
- $[99] \in \{P_1\} = 1$, choosing 99 will allow the next player to choose 100.
- $[98] \in \{P_2\} = 2$, choosing 98 will allow the next player to choose 99 or 100, which have Grundy-values of 1 and 0, respectively. Therefore the smallest number not assigned to a follower is 2.

- $[97] \in \{P_3\} = 3$, choosing 97 will allow the next player to choose 98, 99 or 100, which have Grundy-values of 2,1 and 0, respectively. Therefore the smallest number not assigned to a follower is 3.

- This process repeats until we reach the number 89.

- $[89] \in \{P_{12}\} = 0$, since choosing 89 allows the next player to choose any position left in the game except 100, which has a Grundy-value of 0. Thus the smallest number not assigned to a follower is 0.

It follows that if player A chooses a position with a Grundy-value of 0, then player B goes next and must choose a follower, which cannot have a Grundy-value of 0. Therefore the Grundy-value of player B's position is non-zero, which means that it must have a follower that has a Grundy-value of 0, which can be chosen by player A. One can infer that if A can always choose a position with a Grundy-value of 0, then A can choose $\{P_0\}$=0 and terminate the game.

This resembles a game of Nim in that if player A can get a balanced position, then player B will always be forced to get an unbalanced position. Note that a balanced position equivalent to a safe combination, so when the number of counters in the piles are added vertically in binary modulo 2, the sum is 0. The winning strategy by the Sprague-Grundy theorem is to choose positions with a Grundy-value also of 0. The Grundy-value of a position is equal to the decimal conversion of Nim-value in a game of Nim [5]. Therefore the strategy in an arbitrary normal impartial game and Nim is equivalent.

We can analyze each position in the 100 game to see why $100 - 11n$, $n = 0, 1, 2, \ldots$ gives the safe numbers.This analysis can be seen in the following table.

Table 3.4

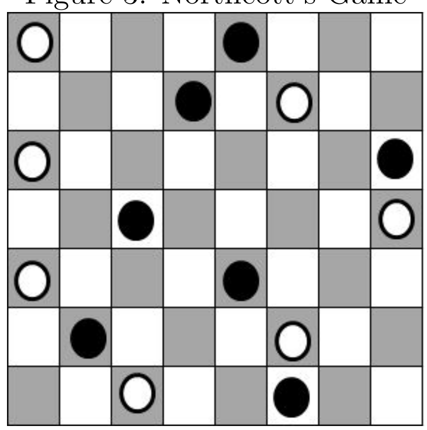
$[100] \in \{P_0\} = 0$	$[67] \in \{P_{33}\} = 0$	$[34] \in \{P_{66}\} = 0$
$[99] \in \{P_1\} = 1$	$[66] \in \{P_{34}\} = 1$	$[33] \in \{P_{67}\} = 1$
$[98] \in \{P_2\} = 2$	$[65] \in \{P_{35}\} = 2$	$[32] \in \{P_{68}\} = 2$
$[97] \in \{P_3\} = 3$	$[64] \in \{P_{36}\} = 3$	$[31] \in \{P_{69}\} = 3$
$[96] \in \{P_4\} = 4$	$[63] \in \{P_{37}\} = 4$	$[30] \in \{P_{70}\} = 4$
$[95] \in \{P_5\} = 5$	$[62] \in \{P_{38}\} = 5$	$[29] \in \{P_{71}\} = 5$
$[94] \in \{P_6\} = 6$	$[61] \in \{P_{39}\} = 6$	$[28] \in \{P_{72}\} = 6$
$[93] \in \{P_7\} = 7$	$[60] \in \{P_{40}\} = 7$	$[27] \in \{P_{73}\} = 7$
$[92] \in \{P_8\} = 8$	$[59] \in \{P_{41}\} = 8$	$[26] \in \{P_{74}\} = 8$
$[91] \in \{P_9\} = 9$	$[58] \in \{P_{42}\} = 9$	$[25] \in \{P_{75}\} = 9$
$[90] \in \{P_{10}\} = 10$	$[57] \in \{P_{43}\} = 10$	$[24] \in \{P_{76}\} = 10$
$[89] \in \{P_{11}\} = 0$	$[56] \in \{P_{44}\} = 0$	$[23] \in \{P_{77}\} = 0$
$[88] \in \{P_{12}\} = 1$	$[55] \in \{P_{45}\} = 1$	$[22] \in \{P_{78}\} = 1$
$[87] \in \{P_{13}\} = 2$	$[54] \in \{P_{46}\} = 2$	$[21] \in \{P_{79}\} = 2$
$[86] \in \{P_{14}\} = 3$	$[53] \in \{P_{47}\} = 3$	$[20] \in \{P_{80}\} = 3$
$[85] \in \{P_{15}\} = 4$	$[52] \in \{P_{48}\} = 4$	$[19] \in \{P_{81}\} = 4$
$[84] \in \{P_{16}\} = 5$	$[51] \in \{P_{49}\} = 5$	$[18] \in \{P_{82}\} = 5$
$[83] \in \{P_{17}\} = 6$	$[50] \in \{P_{50}\} = 6$	$[17] \in \{P_{83}\} = 6$
$[82] \in \{P_{18}\} = 7$	$[49] \in \{P_{51}\} = 7$	$[16] \in \{P_{84}\} = 7$
$[81] \in \{P_{19}\} = 8$	$[48] \in \{P_{52}\} = 8$	$[15] \in \{P_{85}\} = 8$
$[80] \in \{P_{20}\} = 9$	$[47] \in \{P_{53}\} = 9$	$[14] \in \{P_{86}\} = 9$
$[79] \in \{P_{21}\} = 10$	$[46] \in \{P_{54}\} = 10$	$[13] \in \{P_{87}\} = 10$
$[78] \in \{P_{22}\} = 0$	$[45] \in \{P_{55}\} = 0$	$[12] \in \{P_{88}\} = 0$
$[77] \in \{P_{23}\} = 1$	$[44] \in \{P_{56}\} = 1$	$[11] \in \{P_{89}\} = 1$
$[76] \in \{P_{24}\} = 2$	$[43] \in \{P_{57}\} = 2$	$[10] \in \{P_{90}\} = 2$
$[75] \in \{P_{25}\} = 3$	$[42] \in \{P_{58}\} = 3$	$[9] \in \{P_{91}\} = 3$
$[74] \in \{P_{26}\} = 4$	$[41] \in \{P_{59}\} = 4$	$[8] \in \{P_{92}\} = 4$
$[73] \in \{P_{27}\} = 5$	$[40] \in \{P_{60}\} = 5$	$[7] \in \{P_{93}\} = 5$
$[72] \in \{P_{28}\} = 6$	$[39] \in \{P_{61}\} = 6$	$[6] \in \{P_{94}\} = 6$
$[71] \in \{P_{29}\} = 7$	$[38] \in \{P_{62}\} = 7$	$[5] \in \{P_{95}\} = 7$
$[70] \in \{P_{30}\} = 8$	$[37] \in \{P_{63}\} = 8$	$[4] \in \{P_{96}\} = 8$
$[69] \in \{P_{31}\} = 9$	$[36] \in \{P_{64}\} = 9$	$[3] \in \{P_{97}\} = 9$
$[68] \in \{P_{32}\} = 10$	$[35] \in \{P_{65}\} = 10$	$[2] \in \{P_{98}\} = 10$
		$[1] \in \{P_{99}\} = 0$

4 Similiar Games

4.1 Northcott's Game

Northcott's game is played on a checkerboard with exactly one white tile and one black tile per row citeGM. Players, white and black, take turns moving a single piece left or right within its row, without jumping over an opponents piece. The game board resembles figure 3. If a player cannot move, he loses.

Figure 3: Northcott's Game

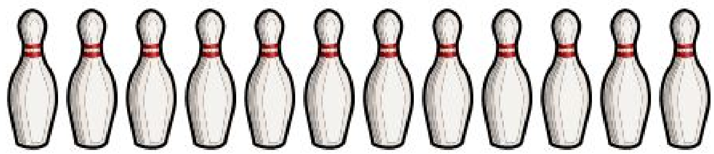


The solution to this game is rather simple. The number of spaces between two tiles within a row is the number of counters in a nim-heap. In figure 3, beginning with the top row, we have piles of 3,1,6,4,3,3,2. This game a bit more complicated than regular Nim since a player can move away from the other player, essentially adding counters to a nim-heap. However, in this case, a player can move as many spaces towards their opponent within that row, or remove the same amount of counters within that pile to restore the games balance.

4.2 Kayles

Now we move to a game known as Kayles, created by H.E. Dudeney [4]. Kayles is played by arranging n quilles, or bowling pins, in a row, and players alternate taking turns knocking down a single pin OR two adjacent pins [4].

Figure 4: Kayles

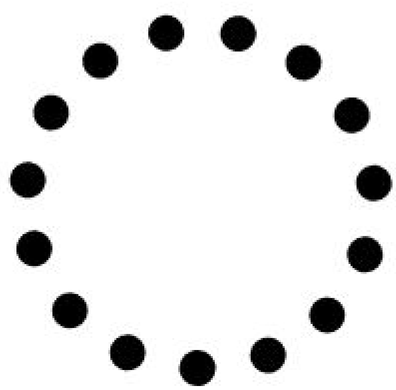


The player to knock down the last pin wins the game. We are to assume that the players are skilled enough to carry out their decision without error.

This game is equivalent to Nim as the pins act as counters in a Nim heap, but the difference is that instead of just removing counters, a player is essentially dividing a pile into two smaller piles if he creates gaps in the row of pins. If the original row is even, a player may knock down the two middle pins, leaving two equal rows or piles, which is obviously a safe combination. Whatever the opponent does to one row, the player can mimic his actions in the other row until there are no pins left. If the row of pins is originally odd, the first player can knock down a single middle pin, leaving two even rows. The similarities between Kayles and Nim is not as hard to see as the 100 Game.

4.3 Circular Nim

Figure 5: Circular Nim



Circular Nim is played by placing counters in a circle, and players take turns removing one counter or two adjacent counters per turn [4]. The player to take the last counter wins the game. This game is very similiar to Nim, after the first few turns. By removing counters, the circle will be broken into smaller segments, which represent Nim heaps. Once the circle is broken into Nim heaps, each player must remove one counter or two adjacent counters but maintain a safe position. This is more difficult since one can divide a pile into two piles by taking counters within the segment. However, since a Nim game is the disjunctive sums of the games of its piles, the strategy of safe positions still holds.

5 Conclusion

I began this research as an attempt to find a solution that I thought did not exist. After dedicated effort and the discovery of a solution to Nim and all variations of the 100 Game, I found to my disappointment that the topic had been thoroughly researched long before my time. The more I learned about the relationships between impartial combinatorial games, I discovered greater connections than I could have imagined. Although many of the results in this paper were previously proven in some way, the process of learning how to make these connections without formal guidance has been more than worthwhile. Impartial combinatorial games are just a small part of the world of game theory, but they are extremely fundamental for the overall understanding of such a broad topic. The logic used to find solutions to these games can't always be applied to real life situations, because they contain no element of chance. The relationship between the games discussed in this paper are astounding, and research on the relationship between general combinatorial games and other games that have disjunctive sums could prove to have greater application. All in all, the world of games and decisions is still full of questions, and I hope this paper was the reader's first step in finding those answers.

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