**Hybrid Musical Instruments: Artistically Inspired, Mathematically Explored**

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# Hybrid Musical Instruments:
# Artistically Inspired, Mathematically Explored

Thesis

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In Fulfilment of the Requirements for
Undergraduate Departmental Honors in
Mathematics

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## Abstract

Musical instruments are intriguing both mathematically and artistically; their forms and the music they produce are beautiful. Thousands of years ago humans recognized the beauty of musical sound but it is only in recent times that scholars have begun to explore the mathematical beauty of music. This thesis is focused on the mathematics of ceramic instruments. This emphasis is inspired by my medium of choice in art, which is clay. Clay is intriguing because of the opportunity to change different variables in each instrument.

There are several questions that this thesis will attempt to answer: What equations govern a family of instruments? How do equations governing typical instruments change with hybrids, instruments that combine aspects from multiple families in one? What do these equations model?

This thesis starts by explaining where math is found in music. Then it states some basic physics equations, their use, how to use them, and what they model. The one-dimensional wave equation, which governs the frequencies of wind and string instruments, is then explored. The solutions are derived and an explanation of their use is given. Important results are explored. The same is done for the two-dimensional wave equation, which governs many percussion instruments. This thesis will also seek to answer the above posed questions for this type of instruments.

# Contents

# Chapter 1

# Mathematics and Music

Math can be found in most aspects of music. Some aspects of music that have been included in mathematical analysis are: harmonics, scales, beats, rhythm, meter, and sound. Once one has a grasp of the mathematics of the above-mentioned components of the music for each class of instruments, exploring the higher order mathematical equations that model hybrid musical instruments is a natural next step in this analysis.

## 1.1: Harmonics, Overtones, and Octaves

Harmonics refers to the doubling, tripling, and other whole number multiples of basic frequencies (Josephs 57). The harmonics found in music from any instrument are frequently analyzed mathematically. Several texts use the word “overtone” in discussing harmonics. An overtone is defined as any higher frequency vibrational mode of a system excluding the fundamental, which is the lowest frequency mode (White 115). There are several overtones, which are produced from an instrument. The first overtone is the second harmonic, the second overtone the third harmonic, and so on.

When a musician moves up an octave, the frequency of the note being played is doubled. When the musician drops an octave, the frequency is halved. For example, the fundamental frequency, or first harmonic, of the note A is 440 Hertz (Hz). Moving up an octave makes the frequency 880 Hz. Mathematically, there is a relationship between certain harmonics and octaves. Moving up one octave doubles the frequency of the first harmonic. Moving up two octaves from the first harmonic implies that one would double the frequency of the first octave. I have created the following theorem, which will allow for ease in calculating the frequency of a given octave.

**Theorem 1.1.1:** If λ is the frequency of a given note, the frequency of a note n octaves higher would be $2^n \lambda$.

Proof by mathematical induction:

Let

$P(n) =$ {If $\lambda$ is the frequency of a given note, the frequency of a note n octaves higher is $2^n \lambda$.}

Then show for $P(1)$. Moving up 1 octave doubles the frequency, so the frequency of a note one octave higher is $2^1 \lambda$. Assume $P(n)$ is true. Then show for $P(n+1)$. The $(n+1)^{th}$ octave is one octave higher than the $n^{th}$ octave. Since $P(n)$ is true, the frequency of the $n^{th}$ octave is $2^n \lambda$. Moving up one octave from $P(n)$ doubles the frequency, which means the frequency of the $(n+1)^{th}$ octave is $2(2^n \lambda) = 2^{n+1} \lambda = P(n+1)$. Hence, by mathematical induction, the statement is true for the set of natural numbers N.

Each harmonic has a corresponding wavelength. The second harmonic has $\frac{1}{2}$ the wavelength of the first. The third harmonic has $\frac{1}{3}$ the length of the original wavelength. This pattern repeats. This relationship, which will be discussed later, leads to the observation that frequency is inversely proportional to the wavelength. In general, waves that have an amplitude and period can be related to sine and cosine waves. If one looks at a general wave where L is the length of the string or air column and the amplitude is taken to be one, the general equation for the wave is $\sin\left(\frac{n\pi}{L}x\right)$ where n is the n^th^ harmonic. This will become more significant as the wave equations are discussed. The following table lists the frequencies of the first eight harmonics of concert A (440 Hz), the corresponding wavelength and sine function.

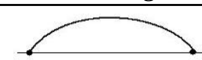
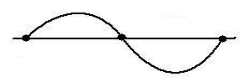
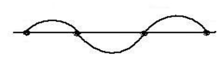
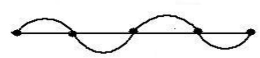
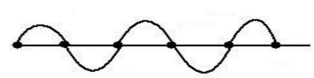
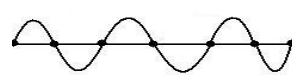
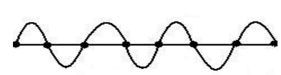
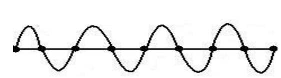
| Harmonic | Octave | Frequency (Hz) | Wavelength | Equation |
|---|---|---|---|---|
| 1 | | 440 |  | $\sin\left(\frac{\pi}{L}x\right)$ |
| 2 | 1 | 880 |  | $\sin\left(\frac{2\pi}{L}x\right)$ |
| 3 | | 1320 |  | $\sin\left(\frac{3\pi}{L}x\right)$ |
| 4 | 2 | 1760 |  | $\sin\left(\frac{4\pi}{L}x\right)$ |
| 5 | | 2200 |  | $\sin\left(\frac{5\pi}{L}x\right)$ |
| 6 | | 2640 |  | $\sin\left(\frac{6\pi}{L}x\right)$ |
| 7 | | 3080 |  | $\sin\left(\frac{7\pi}{L}x\right)$ |
| 8 | 3 | 3520 |  | $\sin\left(\frac{8\pi}{L}x\right)$ |

**Table 1.1.1** ***First Eight Harmonics for Concert A***

## 1.2 Scales, Beats, and Meter

An octave is the only interval found in all scales (Josephs 78). A scale occurs when a pattern of whole and half steps is formed (Brandt 5). This pattern typically takes on ratios between the notes in the scale. Several scales are common enough to have been named: the chromatic, the pentatonic, the Pythagorean, the just and the tempered.

The chromatic scale moves by half steps from one tone to its octave, twelve steps away (Castellini 136). Since the notes in this scale are equidistant, the musician has the choice of which note they will stop playing. Another equidistant scale is the whole-tone scale (137). It contains six whole steps, which will take it to the next octave.

The pentatonic scale is usually found in music of the world from antiquity. This scale uses five black keys or five white keys of a piano played in any order. Much of Western folk music is played in this scale (Castellini 92).

Pythagoras developed a musical scale during his time (sixth century BCE). The Pythagorean scale was a simple recipe for creating new notes from the old ones. He took the existing ratio and multiplied or divided by $\frac{3}{2}$, which is the same as raising or lowering the note by a fifth. Whatever number was gotten from this process was either doubled or halved. If the number was greater than 2, it was halved. If it was smaller than 1, it was doubled. (Johnston 7). Ptolemy expanded upon Pythagoras' theory. Ptolemy's system was expanded to five instead of stopping at four. His work becomes the major third. He found that if you pick a note and raise it by both a third (5:4) and a fifth (3:2), you obtain a major triad (Johnston 12). Pleasing music cannot be created based from math. However, music and musical patterns can be mathematically analyzed.

Pythagoras, a mathematician, based his music theory solely on math.  Today, his 'music' would sound displeasing to most.

The just scale, or intonation, uses whole number ratios between frequencies on the scale (Benton 173).  This scale is a combination of seven whole and half steps (96).  It was noticed that certain combinations of notes were aurally pleasing when sounded together.  The major scale is formed based upon three notes having a ratio of 4:5:6 (Josephs 81).  The minor scale is also built up on a triad.  The ratio for these notes is 10:12:15.  The major scale shall now be derived.  The following table lists the ratios between notes in a given octave starting at C, which has a frequency of 264 Hz.

| **Note** | C | D | E | F | G | A | B | C |
|---|---|---|---|---|---|---|---|---|
| **Ratio** | 1 | 9/8 | 5/4 | 4/3 | 3/2 | 5/3 | 15/8 | 2/1 |
| **Decimal** | 1.000 | 1.125 | 1.250 | 1.333 | 1.500 | 1.667 | 1.875 | 2.000 |

**Table 1.2.1** ***Note Ratios***

These ratios are used to calculate the frequencies of the notes in the scale built from one of the triads.  If the note starting the scale is different from C, the same ratios will be used for each of subsequent notes.

| | C Major | D Major | E Major |
|---|---|---|---|
| $C_4$ | 264 | | |
| D | 297 | 297 | |
| E | 330 | $334^+$ | 330 |
| F | 352 | $371^+$ | 372 |
| G | 396 | 396 | $413^+$ |
| A | 440 | $445^+$ | 440 |
| B | 495 | 495 | 495 |
| $C_5$ | 528 | $557^+$ | $550^+$ |
| $D_5$ | 594 | 594 | $618^+$ |
| $E_5$ | 658 | $668^+$ | 658 |

**Table 1.2.2:** ***Just Major Scale***

Notice that some of the frequencies are superscripted with a $+$. This denotes a frequency, which has been rounded up or down to the appropriate note. This rounding is done when a note in the major to be played cannot be reached perfectly tuned to a different major. Consider a musician who needs to play a note having a frequency of 445 Hz in the D Major. He inconveniently has an instrument tuned to the C Major scale. He arrives at an appropriate solution by playing the note in C Major closest to the frequency to be played, A in this case. The same follows for the other noted frequencies. This adjustment gives rise to the equal tempered scale.

The equal tempered scale is formed by equally dividing the octave. While the other scales account for a specific set of notes, the equal tempered scale allows for a greater variety in a musical piece. The change from eight notes in an octave of the just scale to twelve in the tempered scale forces the ratio value to be recalculated. This change can be mathematically interpreted. One needs to find the ratio between each of the notes. Since the notes are equally spaced, the ratio is whatever number times itself twelve times equals two, for the note in the next octave, or

$$n \times n \times n \times n \times n \times n \times n \times n \times n \times n \times n \times n = n^{12} = 2.$$

Observe.

$$\begin{aligned} &n^{12} = 2 \\ &\sqrt[12]{n^{12}} = \sqrt[12]{2} \\ &n = \sqrt[12]{2} = 2^{\frac{1}{12}} = 1.05946 \end{aligned}$$

So the ratio between each note is 1.059. The table below lists the frequencies for the twelve notes in the octave starting at C

.

| Note | Frequency (Hz) |
|---|---|
| C | 264 |
| C#, Dь | 280 |
| D | 296 |
| D#, Eь | 314 |
| E | 332 |
| F | 352 |
| F#, Gь | 372 |
| G | 394 |
| G#, Aь | 418 |
| A | 442 |
| A#, Bь | 468 |
| B | 496 |
| C | 525 |

**Table 1.2.3:** ***Equal Tempered Scale***

When two instruments play different scales together, one notices a beat, or pattern, of sounds forming. Beats occur when two waves with slightly differing frequencies travel in the same medium (Josephs 23). The pitch that is perceived, rather than being one note or the other, is a combination of the two. The beat frequency is found by the difference of the frequencies, $f_{BF} = f_2 - f_1$. The pitch perceived is found using $f = \frac{f_1 + f_2}{2}$ (White 58).

Meter is a pattern of accented and unaccented beats. There are three common meters: duple, triple, and quadruple. The duple has an accented down beat followed an unaccented beat which is repeated. The triple meter has an accented down beat followed by two unaccented, and the quadruple meter has an accented down beat followed by three unaccented beats (Pen 25).

Now having some knowledge of the mathematical aspects of music, the basics of sound can be explored and a basic vocabulary gained. Sound waves emitted from instruments are longitudinal in nature, moving in the same direction as the medium, air, in which they are traveling. Sine functions are used to describe the motion of sound waves. This is due to the fact that a sound wave has an amplitude, a period, and a basic sine (or cosine) wave, $A\sin(Bx + D)$, which is built from a graph using that knowledge. Amplitude of a transverse wave is the distance from the reference line to the highest or lowest point on the wave. On a longitudinal wave, the amplitude is the maximum displacement of a point from its equilibrium position (Hasting 10). The period is the amount of time in seconds required to make one complete vibration. The period can also be found if the frequency, number of vibrations per second, is known using the formula $T = \frac{1}{f}$.

While the formulas above govern the happenings within an instrument to the simplest degree, others go into much more depth and are more accurate. The wave equation, $u_{tt} - cu_{xx} = 0$, will be used for further exploration of each family of instruments presented in this paper.

# Chapter 2

# Classes of Musical Instruments and the Differential Equations Governing Their Frequencies

## 2.1: Winds

The aerophone, or wind, family of instruments obtained its name from the way it is sounded: vibrating a column of air by blowing wind over or through the vibrating element (Pen). There are two sides to the wind family: the woodwinds and the brasses. The woodwinds, which are sounded when wind is blown across a mouthpiece with either one or two reeds, have four subtypes of instruments: flutes, flageolet, single-reed, and double-reeds. Brass instruments produce sound created by the vibration of a musician's buzzing lips and embouchure, the placement of lips, teeth, and tongue. Flutes have a hole, which the musician blows across, setting the air molecules into vibration as well as the walls of the instrument. Flageolets are designed for the musician to blow into a whistle type mouthpiece that directs the air at a beveled lip. Both single and double reed instruments are sounded by a musician blowing over thin rectangles of cane vibrating against each other.

There are several general observations based on the physical attributes and variables in wind instruments. The pitch of the woodwinds is inversely proportional to the length of the vibrating column of air. The longer the pipe, the lower the note. Likewise, the shorter the length of pipe is, the higher the note. Also, because the tube is thinner than it is long, the note is fairly independent of the diameter. The note also drops an octave when the end of the pipe is closed (Johnston 42). When the pipe is closed, the waves behave as one half of the air column, which is twice as long as the tube. Along the same thought, the shape of the pipe, whether cylindrical or conical, hardly affects the pitch. The air column within the pipe behaves like a stretched string, without boundary conditions, moving along the tube. In a flute, the waves in the column of air move along the pipe. In reed instruments, the reed vibrates, opening and closing due to the force of air, which produces sound. The opening and closing is called the Bernoulli effect. This also occurs in side blown flutes, closed whistles, and ocarinas where air, not a reed, is the cause of the effect.

The type of pipe producing the sound, either closed or open, affects the air column within the instrument and hence the note the ear hears (Josephs). For any pipe, there are several known properties: a node[1] always occurs at the closed end while a loop[2] forms at the open end, the air at the open end always vibrates with maximum displacement equal to the amplitude of the wave, and the column of air of a closed pipe resonates with odd harmonics. The reason the closed pipe resonates with odd harmonics is due to the fact that the closed end forces the displacement to zero (Benson 114). One can see the illustration in the two images below.

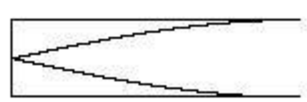


**Image 2.1.1 Displacement of Fundamental**

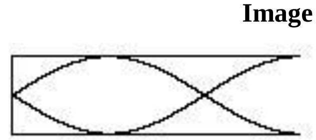


**Image 2.1.2 Displacement of Second Harmonic**

[1] A node is a point along a standing wave where the wave has minimal amplitude.
[2] A loop is the part of a standing wave between the nodes.

Notice that the horizontal displacement of the air column follows the sine waves illustrated in Table 1.1.1.

In mathematics there is a well-known differential equation called the wave equation. It turns out that the function describing the displacement of the air at position x and time t and the function governing acoustic pressure both satisfy the wave equation. This section will show how the functions satisfy the wave equation.

To do this, variables for displacement, $\xi$, and acoustic pressure, p, are introduced (Benson 99). Let x be the position along the tube and $\xi(x,t)$ be the displacement of air at position x at time t. The rest value of pressure is the ambient air pressure $\rho$. Acoustic pressure $p(x,t)$ is measured by subtracting $\rho$ from the absolute pressure[3], $P(x,t)$ so that $p(x,t) = P(x,t) - \rho$.

Hooke's law states

$$p = -B\frac{\partial \xi}{\partial x} \qquad (2.1.1)$$

where B is the bulk modulus[4] of air. Newton's second law implies

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial^2 \xi}{\partial t^2} \qquad (2.1.2).$$

Now use equations 2.1.1 and 2.1.2 to derive a differential equation in terms of the displacement variable $\xi$. Using Equation 2.1.1

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[-B\frac{\partial \xi}{\partial x}\right] = -B\frac{\partial}{\partial x}\frac{\partial \xi}{\partial x} = -B\frac{\partial^2 \xi}{\partial x^2}$$

Substituting this result into Equation 2.1.2, one arrives at

---

[3] The absolute pressure of a system is the total pressure of the system; the sum of the acoustic and ambient air pressures.
[4] The bulk modulus of air is defined as the ability to resist uniform compression.

$$-B\frac{\partial^2\xi}{\partial x^2} = -\rho\frac{\partial^2\xi}{\partial t^2}$$

Rearranging this equation

$$\frac{\partial^2\xi}{\partial t^2} = \frac{B}{\rho}\frac{\partial^2\xi}{\partial x^2}$$

Letting $c = \sqrt{\frac{B}{\rho}}$ one obtains

$$\frac{\partial^2\xi}{\partial t^2} - c^2\frac{\partial^2\xi}{\partial x^2} = 0$$

This is the standard wave equation for displacement.

It is now time to derive a differential equation in terms of the acoustic pressure, $p$. In order to do this, one will take the partial derivative of Equation 2.1.1 with respect to t. This gives

$$\frac{\partial}{\partial t}p = \frac{\partial}{\partial t}\left[-B\frac{\partial\xi}{\partial x}\right] = -B\frac{\partial^2\xi}{\partial t\partial x} \qquad (2.1.3)$$

Since $\xi$ is twice continuously differentiable, its mixed partials are equal. So

$$\frac{\partial p}{\partial t} = -B\frac{\partial^2\xi}{\partial t\partial x} = -B\frac{\partial}{\partial x}\left[\frac{\partial\xi}{\partial t}\right] \qquad (2.1.4)$$

I then took the partial derivative with respect to t of both sides of the previous equation. Since the partial derivative with respect to t of the acoustic pressure is twice continuously differentiable the mixed partial derivatives are equal, which gives way to the following:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial t}\right) &= \frac{\partial}{\partial t}\left(-B\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right)\right) \\
&= -B\frac{\partial^2}{\partial t \partial x}\left[\frac{\partial \xi}{\partial t}\right] \\
&= -B\frac{\partial^2}{\partial x \partial t}\left[\frac{\partial \xi}{\partial t}\right] \\
&= -B\frac{\partial}{\partial x}\left[\frac{\partial^2 \xi}{\partial t^2}\right] \\
\frac{\partial^2 p}{\partial t^2} &= -B\frac{\partial}{\partial x}\left[\frac{\partial^2 \xi}{\partial t^2}\right]
\end{aligned}$$

Substituting Equation 2.1.2 gives

$$\begin{aligned}
\frac{\partial^2 p}{\partial t^2} &= -B\frac{\partial}{\partial x}\left[\frac{\partial^2 \xi}{\partial t^2}\right] \\
&= -B\frac{\partial}{\partial x}\left[\frac{-1}{\rho}\frac{\partial p}{\partial x}\right] \\
\frac{\partial^2 p}{\partial t^2} &= \frac{B}{\rho}\left[\frac{\partial^2 p}{\partial x^2}\right]
\end{aligned}$$

Which becomes

$$\frac{\partial^2 p}{\partial t^2} - \frac{B}{\rho}\left[\frac{\partial^2 p}{\partial x^2}\right] = 0$$

Letting $c = \sqrt{\frac{B}{\rho}}$ gives

$$\frac{\partial^2 p}{\partial t^2} - c^2\frac{\partial^2 p}{\partial x^2} = 0$$

This is the standard wave equation for acoustic pressure.

Observe that the wave equation is a linear combination of the second derivatives: $u_{tt} - cu_{xx} = 0$. As previously stated, sound waves are modeled by sine and cosine equations. This means that one should be able to find the solutions using

$$u = \sin(ct \pm x) \tag{2.1.5}$$

and

$$u = \cos(ct \pm x) \qquad (2.1.6).$$

Consider first $u = \sin(ct + x)$.

$$u_t = c\cos(ct + x), \quad u_x = \cos(ct + x)$$

$$u_{tt} = -c^2 \sin(ct + x), \quad u_{xx} = -\sin(ct + x)$$

$$u_{tt} - c^2 u_{xx} = -c^2 \sin(ct + x) - c^2(-\sin(ct + x)) = 0$$

Observe for $u = \sin(ct - x)$.

$$u_t = c\cos(ct - x), \quad u_x = -\cos(ct - x)$$

$$u_{tt} = -c^2 \sin(ct - x), \quad u_{xx} = -\sin(ct - x)$$

$$u_{tt} - c^2 u_{xx} = -c^2 \sin(ct - x) - c^2(-\sin(ct - x)) = 0$$

Consider now $u = \cos(ct + x)$

$$u_t = -c\sin(ct + x), \quad u_x = -\sin(ct + x)$$

$$u_{tt} = -c^2 \cos(ct + x), \quad u_{xx} = -\cos(ct + x)$$

$$u_{tt} - c^2 u_{xx} = -c^2 \cos(ct + x) - c^2(-\cos(ct + x)) = 0$$

Likewise for $u = \cos(ct - x)$

$$u_t = -c\sin(ct - x), \quad u_x = \sin(ct - x)$$

$$u_{tt} = -c^2 \cos(ct - x), \quad u_{xx} = -\cos(ct - x)$$

$$u_{tt} - c^2 u_{xx} = -c^2 \cos(ct + x) - c^2(-\cos(ct + x)) = 0$$

Hence, it has been shown that $\sin(ct \pm x)$ and $\cos(ct \pm x)$ are solutions to the wave equation as it governs air columns.

## 2.2: Strings

The chordophones, more commonly known as the string family, consists of three different types of instruments: instruments with open strings, fretted instruments, and bowed instruments (Johnston). The family has several universal features that denote an instrument as being a member: attachment devices to attach both ends of the strings to the body of the instrument, a shell that allows vibration to resonate and amplify the sound, and a fingerboard under the string to control pitch (Pen).

There are three methods for setting a string in motion: striking, plucking, or bowing. The wave equation for strings,

$$u_{tt} - c^2 u_{xx} = 0$$

mathematically models the position of the string set into vibration (Sabitova). For each method of setting a string in vibration, the formula changes slightly.

To understand the wave equation for strings, one must first understand the laws modeling the frequency of stringed instruments. Pere Mersenne and Galileo Galilei discovered these laws individually around 1635 (Joseph). They found that the frequency of a stretched vibrating string to be modeled by

$$f = \frac{C}{Ld}\sqrt{\frac{F}{\rho}}$$

where C is the constant of proportionality, L is the length of the string, d is the diameter of the string, ρ is the string's density, and F is the force acting on the string.

Stuart's Calculus book presents a problem that should help one better grasp how the frequency of a note produced from a string changes. It considers what happens as the length, tension, and linear density are changed. The equation that is to be worked with is

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \qquad (2.2.1)$$

Length is the first variable to be explored. Taking the derivative with respect to length gives

$$\frac{\partial f}{\partial L} = -\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}\,.$$

Since the derivative is negative, the frequency is decreasing as the length increases. This means that the longer the string becomes, the smaller the frequency, and hence the deeper the note. Similarly, as the string is shortened, the frequency increases and the note becomes shriller.

I next considered equation 2.2.1 with respect to tension, $T$. The derivative is

$$\frac{\partial f}{\partial T} = \frac{1}{4L}\sqrt{\frac{1}{T\rho}}\,.$$

Since the derivative is positive, the frequency is increasing as the tension is increased, and hence, the note becomes higher. Likewise, as the tension is decreased, so is the frequency and hence the note, becomes lower.

I last considered equation 2.2.1 with respect to the linear density. The derivative with respect to linear density is

$$\frac{\partial f}{\partial \rho} = -\frac{1}{4L}\sqrt{\frac{T}{\rho^3}}\,.$$

Since the derivative is negative, as the density of the string is increased, the frequency slows, and hence the note becomes lower. Similarly, as the string's density is decreased, the frequency increases and the note becomes higher.

Waves in strings travel along the string to its end and back to the other. The waves set into vibration the molecules in the air as well as in another part of the instrument that vibrates more molecules.

A column of air can be set into vibration quite easily. It is not necessary to restrict, or bound, the column of air unless one is attempting to achieve a particular sound. A string however, must be bound in order to vibrate. The length between the bounds can be varied pressing the string at different points which changes the length of the string allowed to vibrate. Changing the length changes the note produced from the vibration.

Consider a string instrument that is set into motion by being struck. An example of an instrument of this nature is a piano. Since the string is fixed at both ends, it gives the following boundary conditions:

$$u(0,t) = u(L,t) = 0$$

where $u(x,t)$ is a solution to the wave equation, x is the horizontal position along the string at time t. Before the string is struck, it is at rest so there are the following initial conditions:

$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

Combing the initial and boundary conditions with the wave equation gives the following partial differential equation system:

$$
\begin{aligned}
&u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \\
&u(0,t) = u(L,t) = 0 \qquad\qquad (2.2.2) \\
&u(x,0) = 0 \\
&u_t(x,0) = 0
\end{aligned}
$$

To solve this system I used the method of separation of variables. I let $u(x,t) = f(x)g(t)$ where $f(x)$ is the function governing the position of the string and $g(t)$ is the function governing the amount of time that had passed.

Observe $u_t = fg_t$ and $u_x = f_x g$. Taking the second derivatives gives $u_{tt} = fg_{tt}$ and $u_{xx} = f_{xx}g$. This gives

$$fg_{tt} - c^2 f_{xx} g = 0$$

which can be rearranged as

$$\frac{f_{xx}}{f} = \frac{1}{c^2}\frac{g_{tt}}{g}.$$

Setting this equation equal to a constant, any real number $k$ gives

$$\frac{f_{xx}}{f} = \frac{1}{c^2}\frac{g_{tt}}{g} = k$$

This gives the equations $f_{xx} = fk$ and $g_{tt} = c^2 kg$.

This gives the system

$$
\begin{aligned}
&f_{xx} - kf = 0 \\
&f(0) = f(L) = 0
\end{aligned}
$$

A solution has the form $f(x) = e^{rx}$. This means $f(x) = e^{rx}$ and $f_{xx} = r^2 e^{rx}$.

Substituting this into system 2.2.2 gives $r^2 e^{rx} - ke^{rx} = 0$ which leads to the characteristic equation $r^2 - k = 0$. The characteristic equation has solutions $r = \pm\sqrt{k}$, where $k \ge 0$ is .

Using the boundary condition $f(0)=0$, $f(0)=A+B=0$ in $f(x)=Ae^{\sqrt{k}x}+Be^{-\sqrt{k}x}$ which gives $B=-A$. So

$$f(x)=A(e^{\sqrt{k}x}-e^{-\sqrt{k}x})=2A\sinh(\sqrt{k}L).$$

Since $\sinh(\mathrm{x})=0$ only if x=0, for $f(L)=2A\sinh(\sqrt{k}L)=0$, A is forced to equal zero. This holds for all k≥0 as there are only trivial solutions. Consider now the case when k<0. Let $\mathrm{k}=-\lambda^2$ so $\mathrm{f}_{\mathrm{xx}}+\lambda^2 f=0$ with the boundary conditions f(0)=f(L)=0. This gives $\mathrm{r}^2+\lambda^2=0$ which becomes $\mathrm{r}^2=-\lambda^2$ . The function becomes $\mathrm{f}=\mathrm{A}\cos\lambda\mathrm{x}+\mathrm{B}\sin\lambda\mathrm{x}$. From above, A=0 which gives $\mathrm{f}=\mathrm{B}\sin\lambda\mathrm{x}$. Using the boundary condition $\mathrm{B}\sin\lambda\mathrm{L}=0$. So for nontrivial solutions of the wave equation, $\lambda_n L=n\pi$ or $\lambda_n=\dfrac{n\pi}{L}$. The solution $\lambda_n$ corresponds to the family of waveforms, which are dependent upon the physical setting on the model.

Return to $\mathrm{g}_{\mathrm{tt}}=\mathrm{c}^2\mathrm{kg}$ and solve for g. Since $\lambda_n$ gave waveforms, $\mathrm{g}_{\mathrm{tt}}-\mathrm{kc}^2\mathrm{g}=0$ is representative of the amplitude for each waveform. While solving for f, it was shown that for k≥0 there are no solutions, so only k<0 is considered. This gives $\mathrm{g}_{\mathrm{tt}}+\lambda^2c^2g=0$, with initial conditions g(0)=1 and $g_t(0)=0$. The characteristic function is $\mathrm{r}^2+(\lambda\mathrm{c})^2=0$. The function becomes $\mathrm{g}=\mathrm{A}\cos\lambda\mathrm{ct}+\mathrm{B}\sin\lambda\mathrm{ct}$. Using the condition g(0)=1, it is found that A=1. To obtain B, take the derivative so that $\mathrm{g}_{\mathrm{t}}=-\lambda\mathrm{cA}\sin\lambda\mathrm{ct}+\lambda\mathrm{cB}\cos\lambda\mathrm{ct}$. Observe that $g_t(0)=\lambda cB=0$, so B=0. That gives $\mathrm{g}(\mathrm{t})=\cos\lambda\mathrm{ct}$. From λ above,

$$g(t)=\cos(\frac{n\pi ct}{L}) \qquad (2.2.3).$$

Combining the equations for $u_n(x,t)$, where x refers to f and t refers to g gives

$$u(f,g) = B_n \cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}),$$

where $B_n$ is a constant and n=1,2,3…. This equation models the relationship between waveforms and vibrations. Notice that the simpler the waveform is, the longer the period.

## 2.3: Drums

The idiophones, or percussion, is the family of instruments, where each instrument has one distinct pitch (Pen). Percussion instruments produce sound by their vibrating elements, like the membrane of a drum or the modified plates in bells, being struck, rubbed, shook, or rattled. Membraneophones, or drums, and triangles are typical instruments of this family. The vibrating element of a percussion instrument is most of, if not the entire, instrument itself.

Membraneophones are different from most of the percussion family in that the vibrating element is a membrane, a stretched flexible sheet of animal hide, or a synthetic variation. Sound is produced when a hand or a stick strikes the stretched membrane. When exploring the mathematics of drums, the membrane is assumed to be ideal. That is, the membrane is assumed flexible, uniform, infinitesimally thick, and stretched equally in all directions by a force unaffected by the motion of the membrane (Josephs). The frequency of vibration of the membrane can be found using the equation $f = \frac{2.405}{2\pi r}\sqrt{\frac{\tau}{\sigma}}$ where r is the radius of the membrane, $\tau$ is the tension of the membrane and $\sigma$ is the area density of the membrane.

While the ability to calculate the frequency of vibration in a membrane is useful, one should be able to describe what equation(s) the waves follow at a given position as a drum is struck. A variation of the wave equation is used:

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 u \\ u(\cdot, 0) &= 0 \end{aligned}, \tag{2.3.1}$$

where $\nabla^2 u$ is the Laplacian of u and the symbol $\cdot$ represents the spatial coordinates that are appropriate (either the Cartesian or Polar). I am looking for a solution of the equation above that is in the form,

$$u(\cdot, t) = g(t) f(\cdot). \tag{2.3.2}$$

The process is similar to the method of separation of variables that I used in finding solutions of the wave equation in relation to the string family. Substituting Equation 2.3.2 into 2.3.1 gives

$$g_{tt} f = c^2 g \nabla^2 f$$

Rearranging the equation gives

$$\frac{g_{tt}}{gc^2} = \frac{\nabla^2 f}{f}.$$

Setting each side of the equation separately to $-\lambda^2$ gives

$$\frac{g_{tt}}{gc^2} = -\lambda^2 \tag{2.3.3}$$

and

$$\frac{\nabla^2 f}{f} = -\lambda^2. \tag{2.3.4}$$

The equation

$$\frac{g_{tt}}{gc^2} = -\lambda^2$$

becomes $g_{tt} = -\lambda^2 gc^2$ which can be rewritten $g_{tt} + \lambda^2 gc^2 = 0$. The value for $g_n$ represents the amplitude function of the $n^{th}$ modes of the wave across the membrane. Since the drum is initially at rest $g_n(0)=0$ and following the steps in solving for $f$ in the struck string problem, $g_n = c_n \sin c\lambda_n t$. Taking $c_n$ equal to one, $u(\cdot,t) = \sin(c\lambda_n t) f(\cdot)$.

Look now at (2.3.4). The equation

$$\nabla^2 f = -\lambda^2 f$$

becomes

$$\nabla^2 f + \lambda^2 f = 0.$$

A working equation to use for the Laplacian is needed. Consider a circular membrane of unit radius. Take

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f).$$

That is the same as

$$\nabla \cdot < f_x, f_y >$$

which becomes

$$f_{xx} + f_{yy}.$$

Since a circular membrane is being considered, it makes sense that that variables are changed from Cartesian coordinates to their polar coordinate equivalent. To do this take $\mathrm{x} = \mathrm{r}\cos\theta$ and $\mathrm{y} = \mathrm{r}\sin\theta$. Take the derivatives with respect to *r* first. The first derivative will be of the form

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}.$$

For consistency the notation $f_r = f_x x_r + f_y y_r$ will be used. The derivative becomes

$$f_r = \cos\theta \ f_x + \sin\theta \ f_y.$$

The second derivative is found to be

$$f_{rr} = \lfloor \cos\theta \ f_x + \sin\theta \ f_y \rfloor_r.$$

$f_{rr}$ can be rewritten as

$$\cos\theta \lfloor f_{xx} x_r + f_{xy} y_r \rfloor + \sin\theta \lfloor f_{yx} x_r + f_{yy} y_r \rfloor$$

which, when simplified, becomes

$$\cos\theta \left( f_{xx} \cos\theta + f_{xy} \sin\theta \right) + \sin\theta \left( f_{yx} \cos\theta + f_{yy} \sin\theta \right).$$

So

$$f_{rr} = \cos^2\theta \ f_{xx} + 2\sin\theta\cos\theta \ f_{xy} + \sin^2\theta \ f_{yy} \qquad (2.3.5).$$

Now take the derivatives with respect to θ. So $f_\theta = -r\sin\theta f_x + r\cos\theta f_y$. That means

$f_{\theta\theta} = [-r\sin\theta \ f_x + r\cos\theta \ f_y]_\theta$ which is written

$$f_{\theta\theta} = -r\cos\theta \ f_x - r\sin\theta(f_{xx} x_\theta + f_{xy} y_\theta) - r\sin\theta \ f_y + r\cos\theta(f_{yx} x_\theta + f_{yy} y_\theta).$$

Continue to simplify so that

$$f_{\theta\theta} = -r(\cos\theta \ f_x + \sin\theta \ f_y) - 2r^2 \sin\theta\cos\theta \ f_{xy} + r^2(\sin^2\theta \ f_{xx} + \cos^2\theta \ f_{yy}) \quad (2.3.6).$$

Return to the eigenvalue problem from above:

$$\nabla^2 f + \lambda^2 f = 0.$$

Ledder rewrites the equation as

$$\frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r^2} f_{\theta\theta} + \lambda^2 f = 0 \qquad (2.3.7)$$

(535). I would like to show that

$$\frac{1}{r}\frac{\partial}{\partial r}(rf_r)+\frac{1}{r^2}f_{\theta\theta} = f_{xx} + f_{yy} = \nabla^2 f \qquad (2.3.8).$$

Observe that

$$\frac{1}{r}\frac{\partial}{\partial r}(rf_r) = \frac{1}{r}\left[f_r + rf_{rr}\right] = \frac{1}{r}f_r + f_{rr} \qquad (2.3.9)$$

Substituting in the appropriate above equations (2.3.5) and (2.3.6) for

$$f_{rr} + \frac{1}{r^2}f_{\theta\theta} \qquad (2.3.10)$$

yields

$$(\sin^2\theta+\cos^2\theta)f_{xx}+(\sin^2\theta+\cos^2\theta)f_{yy}+2\sin\theta\cos\theta\, f_{xy}-2\sin\theta\cos\theta\, f_{xy}-\frac{1}{r}(\cos\theta\, f_x+\sin\theta\, f_y)$$

. Which, using trigonometric identities[5], reduces to

$$f_{xx} + f_{yy} - \frac{1}{r}f_r.$$

This means that

$$f_{rr} + \frac{1}{r^2}f_{\theta\theta} + \frac{1}{r}f_r = f_{xx} + f_{yy} = \nabla^2 f\ . \qquad (2.3.11)$$

The left side of the equation can be rewritten as

$$\frac{1}{r}\left[f_r + rf_{rr}\right] + \frac{1}{r^2}f_{\theta\theta}.$$

One continues to reduce to

$$\frac{1}{r}f_r + f_{rr} + \frac{1}{r^2}f_{\theta\theta}.$$

Which substituting (2.3.8) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(rf_r)+\frac{1}{r^2}f_{\theta\theta} \qquad (2.3.12)$$

[5] The Pythagorean identity $\cos^2 x+\sin^2 x=1$ was used.

Combining (2.3.7) and (2.3.9) shows (2.3.12) is valid. Return now to the wave equation, which is written as

$$\frac{1}{r}\frac{\partial}{\partial r}(rf_r)+\frac{1}{r^2}f_{\theta\theta}+\lambda^2 f=0\,.$$

To find solutions to the wave equation, one again uses separation of variables. Begin with $\mathrm{f}(\mathrm{r},\theta)=\mathrm{R}(\mathrm{r})\Theta(\theta)$ where $\mathrm{f}_{\mathrm{r}}=\mathrm{R}_{\mathrm{r}}\Theta$. Replacing appropriate values gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rR_r\Theta\right]+\frac{1}{r^2}R\Theta_{\theta\theta}+\lambda^2 R\Theta=0$$

which becomes

$$\frac{1}{r}\left[R_r\Theta+rR_{rr}\Theta\right]+\frac{1}{r^2}R\Theta_{\theta\theta}+\lambda^2 R\Theta=0\,.$$

Distribute so that

$$\frac{1}{r}R_r\Theta+R_{rr}\Theta+\frac{1}{r^2}R\Theta_{\theta\theta}+\lambda^2 R\Theta=0\,.$$

Rearranging the terms so that

$$\frac{1}{r}R_r\Theta+R_{rr}\Theta+\lambda^2 R\Theta=-\frac{1}{r^2}R\Theta_{\theta\theta}$$

and dividing both sides by R gives:

$$\frac{1}{r}\frac{R_r}{R}+\frac{R_{rr}}{R}+\lambda^2=-\frac{1}{r^2}\frac{\Theta_{\theta\theta}}{\Theta}\,.$$

Multiplying through by $r^2$ and setting both sides equal to a constant,

$$r\frac{R_r}{R}+r^2\frac{R_{rr}}{R}+r^2\lambda^2=-\frac{\Theta_{\theta\theta}}{\Theta}=k\,.$$

One now solves for each function R and Θ.

Consider R first. The equation is

$$r\frac{R_r}{R}+r^2\frac{R_{rr}}{R}+r^2\lambda^2=k\,.$$

In setting the equation equal to zero,

$$rR_r r^2 R_{rr}+R(r^2\lambda^2-k)=0\,,$$

a parametric Bessel equation is obtained. There is an implicit boundary condition $|\mathrm{R}(0)|<\infty$. It states that at the center of the drum, where the radius is zero, the amplitude of the wave is less than infinity. R(1)=0 is the other condition. Since the drum being considered has unit radius, the radius equaling one tells us that at the rim of the drum there is no movement. From previous solutions when $\lambda=0$, there are no nonzero solutions and hence all solutions are trivial. Consider $\lambda>0$. The solution becomes

$$R=c_1J_m(\lambda r)+c_2Y_m(\lambda r)$$

where $c_1$, $c_2$ are constants, $J_m$ is the Bessel equation of the initial kind, and $Y_m$ is the Bessel equation of the second kind. Since the drum head is bounded, $c_2Y_m(\lambda r)$ can be ignored as it refers to an unbounded equation.

Turn now to the equation for $\Theta$:

$$-\frac{\Theta_{\theta\theta}}{\Theta}=k\,.$$

Setting the equation equal to zero gives $\Theta_{\theta\theta}+k\Theta=0$. Again consider k>0. To ensure that k>0, let $m=\sqrt{k}$. The equation becomes $\Theta_{\theta\theta}+m^2\Theta=0$. There are two additional conditions to those above: $\Theta(2\pi)=\Theta(0)$ and $\Theta_\theta(2\pi)=\Theta_\theta(0)$. These conditions are important because they require that the membrane on the drum be continuous when the radius equals one. The conditions make sense because $\theta=0$ and $\theta=2\pi$ represent the same point on the membrane. The solution to the differential equation becomes

$$\Theta=a\cos m\theta+b\sin m\theta\,.$$

The total solution to the wave equation as it governs stretched membranes is

$$R \cdot \Theta = c_1 J_m(\lambda r)(a \cos m\theta + b \sin m\theta) \qquad (2.3.13).$$

To help visualize what is happening across a membrane, as it is set into vibration, think of a trampoline with a single jumper. Before the individual climbs on, the trampoline is at rest. Once the jumper is in place at the center, they start to set the trampoline into motion. As the vibrations of the trampoline become greater, the jumper flies higher into the air. Notice that the trampoline only gives so much. With each jump on the trampoline, waves are sent out from the jumper's location and back, much like a musician beating at the center of a drum's membrane with a mallet.

Visualizing what is happening across a membrane played with more than one mallet or both a musicians hands is slightly more difficult. We give the same example of a trampoline, but now with two jumpers. Since neither can jump in the same spot, the waves sent across the trampoline are continually disrupted. The same happens with a drum. Each beat interrupts the wave patterns from the previous beat.

# Chapter 3

# Ceramics

The crux of this paper has been reached. How are the frequencies of a musical instrument changed when constructed from ceramic material? Ideally, the material from which an instrument is made will not change anything. In practice and with a trained ear, the frequencies change between every instrument and with each different construction material.

Before delving farther into the discussion of how clay affects an instrument's frequencies, one needs to know more about the basic properties of clay. There are many types of clay bodies, each with their own properties that are a factor when constructing any piece. A shrinkage table for several different clay bodies is provided. What happens when one type of clay is used to construct flutes will also be explored. The chemical composition of a clay body will affect two things: the amount of shrinkage between each firing and the firing temperature.

Related to the firing temperature of a clay body is the vitrification of the clay. A clay body is vitrified when it is water tight without the use of a glaze. Vitrification is important as the clay body itself, when struck, gives a higher pitched ring. It is thought that the vitrification of the clay body in the instruments will have the same effect.

The percentage a clay piece shrinks is also an issue between different clay bodies. In general, to calculate the shrinkage of a clay body one rolls out a slab of clay of even thickness ten inches long (this makes calculations easier). The slab is then bisque fired and remeasured. Harvey Brody states his process of calculating shrinkage:

“1. Make several bars at least 12 centimeters long and 1 centimeter thick from the wet clay of the consistency you plan to use.

2. Scratch two marks on each bar exactly 100 millimeters (10 centimeters) apart.

3. Dry the bars completely.

4. Measure the distance between the two marks. Each millimeter is 1 percent of the total. If the distance now measures 97 mm after drying, then drying shrinkage was 3 percent.

5. Fire the bars to whatever temperature you are testing them at; when cool, measure the distance again. If it now measures 95 mm, the firing shrinkage was 2 percent and the total shrinkage 5 percent (10).”

Brody’s method is just one way to calculate shrinkage. I use a slightly differing method of calculation:

$$s = 100\left(\frac{C_b - C_a}{C_b}\right),$$

where $C_b$ is the length of clay prior to firing and $C_a$ is the length of clay after firing. Either method allows one to account for shrinkage before a piece is fired. If a slab of a specific size after a firing is desired, the original clay piece is increased by $p = 100 - s$, where $p$ is the percentage of a piece remaining after a shrinkage of $s$ percent:

$$T = \frac{L}{(1 + p)}$$

where $T$ is the size of the piece before firing, $L$ is the size of the piece needed after a firing, and $p$ is that of the paragraph above. Any clay piece can easily go through two or more firings with each firing shrinking the piece. Shrinkage calculations should be done for each clay body between each firing. Eventually an overall percentage of shrinkage will be obtained.

To help understand the issue of shrinkage, I have rolled out a sample slab of each clay body and constructed several side blown flutes of varying lengths from a single clay body. The flutes were extruded as a hollow tube and allowed to firm up overnight. One end was then closed and the mouth and finger holes were made. They were allowed to dry slowly for about a week so they could be bisque fired. The flutes were then decorated so that they were functional and glaze fired. A table showing the calculated shrinkages for the slabs and each length of flute is included below.

| | **Length (inches)** | | **Shrinkage %** |
|---|---|---|---|
| **Clay Type** | **Wet** | **Bisque** | |
| Buffalo Wallow | 10 | 9.25 | 7.5 |
| Raku | 10 | 9.625 | 3.75 |
| Porcelain | 10 | 9.875 | 1.25 |
| Longhorn White | 10 | 9.5 | 5 |
| High Fire White | 10 | 9.875 | 1.25 |
| Studio | 10 | 9.375 | 6.25 |

**Table 3.1** ***Shrinkage Percentages for Clay Bodies***

| **Longhorn White Flutes** | | | | | | |
|---|---|---|---|---|---|---|
| **Flute** | **Length Wet (in)** | **Bisque (in)** | **Shrinkage (%)** | **Glaze (in)** | **Shrinkage (%)** | **Total Shrinkage (%)** |
| I | 14.75 | 14.25 | 3.39 | 14.25 | 0.00 | 3.39 |
| II | 14.75 | 14.00 | 5.08 | 14.00 | 0.00 | 5.08 |
| III | 15.25 | 14.50 | 4.92 | 14.50 | 0.00 | 4.92 |
| IV | 16.25 | 15.50 | 4.62 | 15.38 | 0.81 | 5.38 |
| V | 16.00 | 15.50 | 3.13 | 15.38 | 0.81 | 3.91 |
| VI | 15.50 | 15.00 | 3.23 | 14.75 | 1.67 | 4.84 |
| VII | 13.00 | 12.75 | 1.92 | 11.75 | 7.84 | 9.62 |
| VIII | 12.50 | 12.00 | 4.00 | 11.75 | 2.08 | 6.00 |
| IX | 18.50 | 17.75 | 4.05 | 17.75 | 0.00 | 4.05 |
| X | 20.25 | 19.25 | 4.94 | 19.13 | 0.65 | 5.56 |
| XI | 19.50 | 18.50 | 5.13 | 18.25 | 1.35 | 6.41 |
| XII | 18.75 | 17.75 | 5.33 | 17.75 | 0.00 | 5.33 |
| XIII | 18.75 | 17.75 | 5.33 | 17.50 | 1.41 | 6.67 |
| XIV | 18.75 | 18.00 | 4.00 | 17.75 | 1.39 | 5.33 |
| XV | 20.00 | 19.50 | 2.50 | 19.38 | 0.64 | 3.13 |
| blank | 17.00 | 17.00 | 0.00 | 16.88 | 0.74 | 0.74 |
| | | | | | | |
| | | | **Average Total Shrinkage** | | | 5.02 % |

**Table 3.2** ***Shrinkage Table for Lowfire Flutes***

We turn now to ceramic models of instruments and how they help us understand the math previously explored. The flutes from Table 3.2 will help us hear what the one-dimensional wave equation models. Drums of varying shapes, sizes, and drumheads will help us to hear what the two dimensional wave equation models.

The basic physics equations governing waves will be considered and applied to the flutes. Each flute, because of their varying lengths and varying levels of vitrification, has a different frequency.

Applying physics to the drums will be difficult as the equations governing the waves of a two dimensional surface are much more complicated. The difference in the pitch of a drum is heard much easier than calculated. A deep resonating boom is heard when the oversized drum “Boomer” is struck. A higher pitched thud sounds from the smallest drum.

Various components of ceramic instruments have been discussed in terms of math. What physically happens during the creation of the instruments and what were some of the problems encountered? Not knowing anything about instruments or what really went into instrument construction made the task difficult. As knowledge was gained about various instruments and their construction, the challenge became more about how to incorporate those specific things into a clay counterpart. After the workings were figured out, the next concern was creating an instrument of uniform thickness. Coil built instruments tended to have this problem the most. The coils may have been the same size but blending and shaping the instruments could and did change the thickness without the builder being aware of it. Extruded flutes also struggled with uniform thickness. The die cut used was not always centered and never stayed in the same place.

Construction of instruments was not the only problem. Firing the instruments proved to be problematic. A large instrument, like "Boomer", had to be moved and loaded with two people. It barely fit in the kiln. The flutes had the most problems during firing. Some were too long to fit in the kiln and were therefore fired at angles. Those that could fit were fired on shelves that were bowed. Both ways of firing the flutes caused bowing in the flutes. The bowing continued in the glaze firing. In the future, the dimensions of the kiln will be kept in mind as pieces are built. Glaze firings are not kind to flutes. A flute must be glazed everywhere to be functional. To be fired, the flute must rest on stilts. This can be a problem as flutes can, and have rolled off the stilts and stuck to the kiln shelf. Another glaze problem is that the glaze can clog the airway or finger holes. Aside from glaze issues not allowing the flute to function, there can be other glaze problems that would render a flute dysfunctional. Some problems with glazes are crawling, shivering, under firing, or clogging of finger holes.[6]

# Chapter 4

# Further Research: Hybrid Instruments

Hybrid instruments are instruments that combine key features from two or more families. The concept of a hybrid instrument came from Barry Hall's Mud to Music. His book explores several artists who create ceramic instruments and presents multiple hybrid

[6] Crawling is a glaze defect in which the glaze separates on a piece. This is usually due to wax or oil from the artist's hands. Shivering is a defect where a glaze flakes off a piece.

instruments.   Some of the hybrid instruments presented are “Stone Fiddle,” “Elephonium,” “Globutubular Drum Horn,” and “Ocarina Drum” (Hall). The stone fiddle combines a flute, a drum, and fiddle.  “Elephonium” is part drum, part harp.  The drum horn is just that; a combination of a drum and a horn.  Ocarina drum also names its parents; the ocarina and a drum.

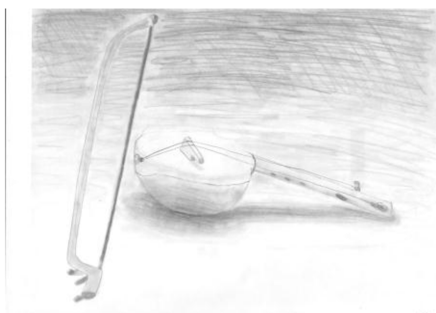
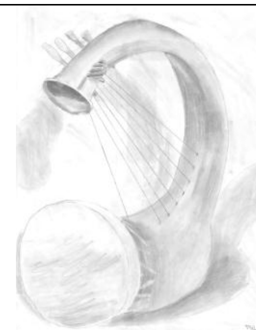
| | |
|---|---|
|  “Stone Fiddle” |  “Elephonium” |
|  “Globutubular Drum Horn” |  “Ocarina Drum” |

**Table 4.1** ***Sketches of Hybrid Instrument***

It is believed that a combination of parts of equations governing other families of instruments will result in an equation modeling the hybrid.  Each hybrid will have a different equation based upon the families the hybrid is composed of as well as how the piece is analyzed.

There are several thoughts about how the equation will be created: as a composition of the different wave equations, the product of the equations, or a simple summation of the values for each section of the family using the respective wave equations.

If, as in a stone fiddle, we look first at the flute and its effects on the drum and string, the initial conditions for them will be different as opposed to if they had been at rest, then set into motion. If we plucked the string then blew into the flute, the membrane's original position is different from its resting position. If the drum is struck first, it will affect the initial conditions for the string.

In the most basic form, it is thought that the equation governing hybrids will be a summation of equations 2.1.5, 2.2.3, and 2.3.13, which looks like

$$\sin(ct \pm x) + \cos(\frac{n\pi ct}{L}) + c_1 J_m(\lambda r)(a\cos m\theta + b\sin m\theta) \qquad (4.1).$$

This paper has looked at and explored areas in music where math can be applied, some basic physics equations governing frequency, and the wave equation as it governs different families of instruments at the most basic level. It also provided a very basic look at some complications and concerns in using clay as a construction material for musical instruments.

In writing the paper, the above pose questions were answered and many more questions were raised. As the topic of equations for hybrids continues, many experiments will be conducted using measuring devices for accuracy. In the future, there may very well be an equation found that can capture each hybrid's mathematical essence.

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