THE TRAVELING SALESWOMEN MEETS SPERNER

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ABSTRACT. In the traveling salesman problem for three cities, it can be shown, with the use of Sperner's Lemma, that there is always a coordinate for homeport where all paths of travel are equal in distance. It is possible to compute this point (the Sperner Point) in the case that the cities form an equilateral or an isosceles triangle. However, when the cities are in a scalene formation, the Sperner Point is not easily computable. Using Newton's Method, a good estimation of the Sperner Point can be found; with the use of the centroid of the triangle formed by the cities to fuel Newton's Method, it is possible to always find a convergent answer.

In the traveling salesman problem, a salesman must leave homeport and visit each city on her route once and only once before returning home. In the case where the salesman needs to visit three cities, she must leave homeport (H) and travel to cities A, B, and C once and only once before returning back to homeport as seen in Figure 1. Our traveling salesman is always looking to minimize her distance traveled. An optimal path for a specific homeport is a path that minimizes the distance traveled from that homeport.

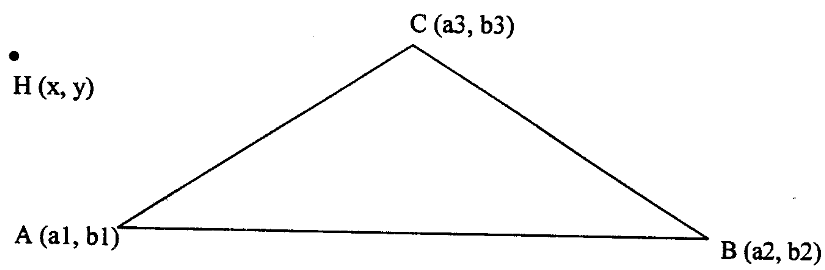

Figure 1: Three Cities

There are three possible paths the salesman can take assuming that traveling the route backwards is the same distance as traveling it forward. (i.e. Home-A-B-C-Home is equivalent to Home-C-B-A-Home).

P1 = H-A-B-C-H

P2 = H-A-C-B-H

P3 = H-B-A-C-H

Depending on where H=(x,y) is located, the paths require different amounts of travel. For the purpose of classifying different positions of homeport, if (x,y) is a point where P1 is optimal, that point shall be colored red. If P2 is the optimal path for (x,y), that point shall be colored blue, and all points where P3 is the optimal shall be colored green.

$$\text{(RED) P1} = \sqrt{(x-a1)^2+(y-b1)^2} + AB + BC + \sqrt{(x-a3)^2+(y-b3)^2}$$

$$\text{(BLUE) P2} = \sqrt{(x-a1)^2+(y-b1)^2} + AC + BC + \sqrt{(x-a2)^2+(y-b2)^2}$$

$$\text{(GREEN) P3} = \sqrt{(x-a2)^2+(y-b2)^2} + AB + AC + \sqrt{(x-a3)^2+(y-b3)^2}$$

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A specific home point could result in one, two, or all three of the paths being optimal. In an unpublished paper[1], Bellino and Narasimhan found that if H is moved and the optimal path is recorded, regions of colors are formed. The boundaries of the colored regions are formed when two paths are optimal (i.e. P1=P2, P2=P3, and P1=P3). Depending on the relative location of the cities, the boundaries are different types of functions. Where all three functions intersect (P1=P2=P3), all paths are optimal. Sperner's Lemma proves this intersection of all three paths will always occur.

SPERNER'S LEMMA:

Consider a combinatorial subtriangulation of triangle ABC, as in Figure 2. Label A, red; B, blue; and C, green. Then label vertices on AB either red or blue, vertices on BC either blue or green, and vertices on AC either red or green. Then label vertices in the interior of ABC red, blue, or green. Then there exists a triangle in the triangulation that contains vertices labeled red, blue, and green.

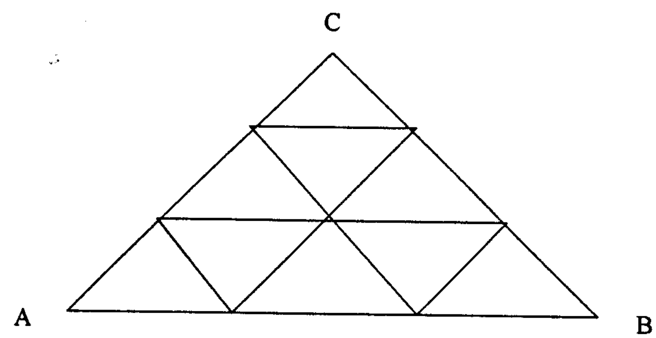


Fig. 2: Sperner's Lemma

In our triangle, ABC, formed by the three cities, suppose homeport was on the edge AB, it would have the same optimal path as a homeport located at either A or B. With this information, it is true that points on AB will be labeled with the same color as either A or B. The same idea follows for edges BC and AB. Since the edges of ABC will always be labeled as described in Sperner's Lemma, there will always be the situation where the traveling salesman can place her homeport on a point where no matter which path she decides to take, the same amount of travel is required. If she decided to build her home on this point she could call it Sperner Point.

We would like to be able to find the Sperner Point no matter where the cities are located.

EQUILATERAL CASE

The easiest case in which to find the Sperner Point is the equilateral triangle. The colored regions are bounded by straight lines as exhibited in Figure 3.

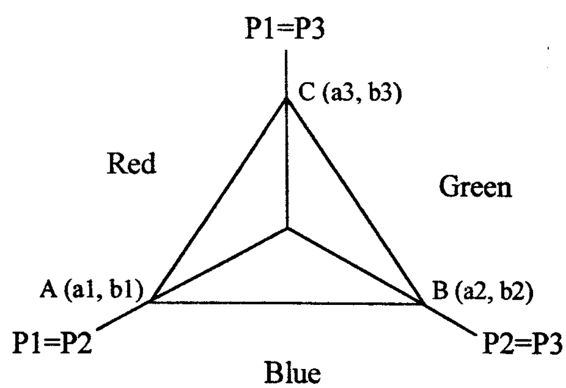


Figure 3 : Equilateral Triangle

The equation for the function separating the areas of red and blue regions would be where P1=P2, i.e.

$$\sqrt{(x-a1)^2+(y-b1)^2}+AB+BC+\sqrt{(x-a3)^2+(y-b3)^2}$$
$$=\sqrt{(x-a1)^2+(y-b1)^2}+AC+BC+\sqrt{(x-a2)^2+(y-b2)^2}.$$

Since all sides are the same length, the resulting equation is

$$\sqrt{(x-a3)^2+(y-b3)^2}=\sqrt{(x-a2)^2+(y-b2)^2}.$$

Simplifying we have

$$(a3-a2)x+(b3-b2)y=a1(a3-a2)+b1(b3-b2).$$

The same result is observed when P1=P3, giving the line

$$(a1-a2)x+(b1-b2)y=a3(a1-a2)+b3(b1-b2)$$

as the red/green boundary, and the function representing P2=P3 is the line

$$(a1-a3)x+(b1-b3)y=a2(a1-a3)+b2(b1-b3).$$

Since all three graphs are straight lines and intersect at some point (Sperner's Lemma), with algebra it is possible to find the intersection of two lines and obtain the Sperner Point.

ISOSCELES CASE

In the case where cities A, B, and C form an isosceles triangle, instead of three lines, the colored regions are separated by two hyperbolas and a line. The distinction between the areas of P1 and P3 being optimal, is denoted by the equation P1=P3. Since AC is equal to BC, the function can be written the same as in the equilateral triangle. However, in the equations P1=P2 and P2=P3, the sides of the triangle involved are not of equal length. The equation P2=P3 can be simplified into the form

$$\sqrt{(x-a1)^2+(y-b1)^2}-\sqrt{(x-a3)^2+(y-b3)^2}=AB-BC.$$

The graph P1=P2 follows the same pattern. So in the case of an isosceles triangle, the Sperner Point is determined by the intersection of one line and two hyperbolas

Consider the following configuration of cities A, B, and C as in Figure 4. Note that any isosceles triangle can be oriented in this manner.

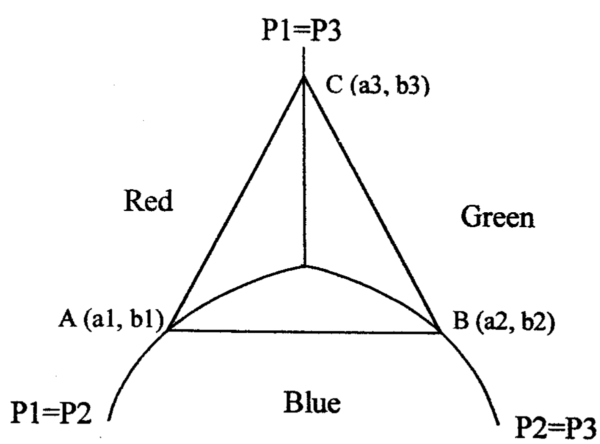


Figure 4 : Isosceles Triangle

The Sperner Point will lie on the line x=a3. So a substitution for the x-value can be made into one of the hyperbolas in order to find the y-value of the Sperner point. So if we begin with the hyperbola given by P2=P3,

$$\sqrt{(x-a1)^2+(y-b1)^2}-\sqrt{(x-a3)^2+(y-b3)^2}=AB-BC,$$

where x=a3, we obtain

$$\sqrt{(a3-a1)^2+(y-b1)^2}-\sqrt{(y-b3)^2}=AB-BC.$$

Solving for y, it is possible to find the y coordinate of the Sperner Point to be

$$y=\frac{(AB-BC)^2-(a3-a1)^2-(b1)^2+2(AB-BC)(b3)+(b3)^2}{2(b3)-2(b1)+2(AB-BC)}.$$

SCALENE CASE

The case of the scalene triangle poses a more difficult problem. No two sides are equal; it is now the case that three hyperbolas are involved. The boundaries of the colored regions are formed by the equations:

P1=P2,

$$\sqrt{(x-a3)^2+(y-b3)^2}-\sqrt{(x-a2)^2+(y-b2)^2}=AC-AB;$$

P2=P3,

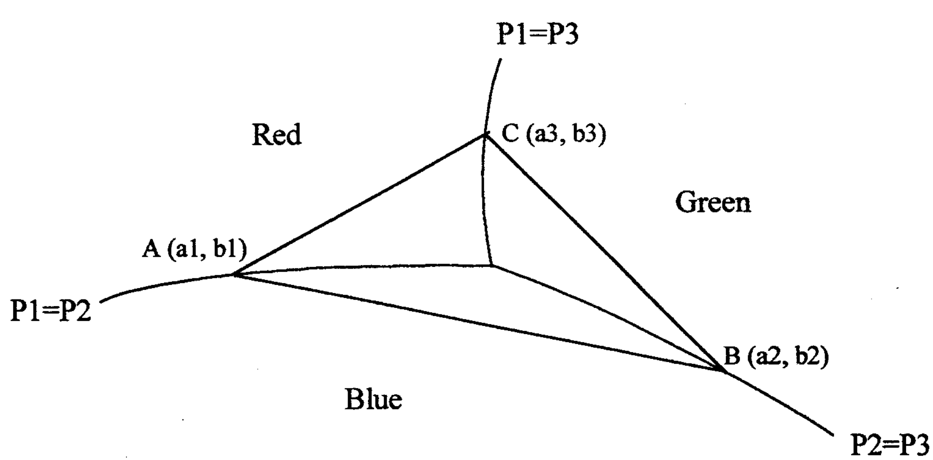
$$\sqrt{(x-a1)^2+(y-b1)^2}-\sqrt{(x-a3)^2+(y-b3)^2}=AB-BC;$$

and P1=P3,

$$\sqrt{(x-a1)^2+(y-b1)^2}-\sqrt{(x-a2)^2+(y-b2)^2}=AC-BC.$$

In order to find the Sperner Point, it is necessary to find the intersection point of two hyperbolas. Since the equations of the hyperbolas contain both the x and y values under the radicals, it becomes difficult to solve for either variable. With such a difficult equation it seems reasonable to find an estimation using Newton's Method.

Newton's Method requires an initial guess. If the initial input is not close enough to the actual root of the problem, the answer may not be convergent. In the case of our traveling salesman, a point in the triangle is needed, which is close enough to the actual Sperner Point, so that Newton's Method will always converge. The centroid was found to be a good starting point.



scalene

DEFINITION OF THE CENTROID:

Given a triangle with vertices A, B, and C, the centroid is the intersection of the lines going from the midpoint of AB to C, the midpoint of AC to B, and the midpoint of BC to A.

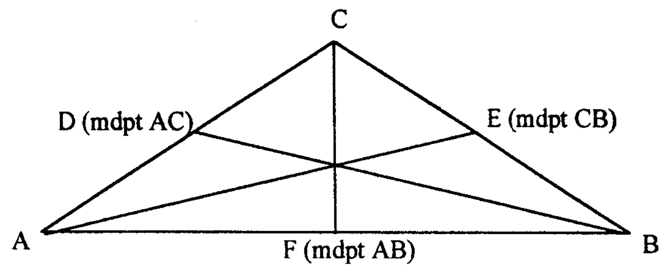


Figure 5: Construction of the Centroid

Since the coordinates of the triangle are known, the centroid is fairly easy to compute. The intersection of BD and AE will give the centroid.

$$AE = y = \frac{b3 + b2 - 2(b1)}{a3 + a2 - 2(a1)}(x - a1) + b1$$

$$BD = y = \frac{b3 + b1 - 2(b2)}{a3 + a1 - 2(a2)}(x - a2) + b2$$

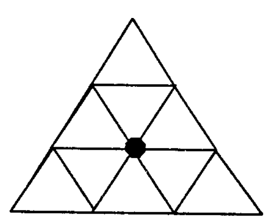
So the intersection of these two lines is

$$\frac{b3 + b2 - 2(b1)}{a3 + a2 - 2(a1)}(x - a1) + b1 = \frac{b3 + b1 - 2(b2)}{a3 + a1 - 2(a2)}(x - a2) + b2.$$

simplification gives the coordinate of the centroid to be the average of the vertices:

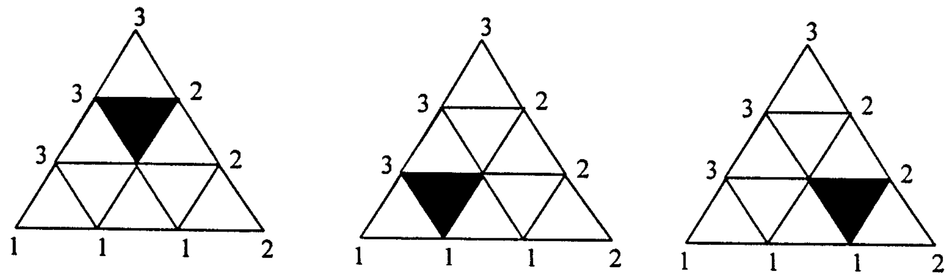
$$(\frac{x1 + x2 + x3}{3}, \frac{y1 + y2 + y3}{3}).$$

In order to get a better idea of how the centroid and the Sperner Point are related, it is important to look at the following way to construct the centroid. If lines are drawn parallel to each base which split the height of the triangle in thirds, the centroid is the intersection of the three lines that segment the lower third.



Construction of Centroid

If the above triangulation is applied to the triangle formed by cities A, B, and C, and the edges of the triangle are labeled as in Sperner's Lemma, the centroid becomes the only point of the triangle which has not yet been colored. The shape of the triangle will indicate the color of the centroid; however, Sperner's Lemma states that no matter what the centroid is labeled, there will be a subtriangle which has all three colors as the vertices. One of three specific subtriangles will always be tricolored. Figure 6 displays the three options.



The Sperner point will be in the triangle which is labeled with all three colors. If the centroid of the original triangle does not give a convergent answer when Newton's Method is applied, it is possible to just work with the subtriangle. All of the subtriangles are similar, thus the triangle containing the Sperner point is one/ninth the area of the original triangle. It is possible to subdivide the tricolored triangle in the same manner as the original triangle so that Sperner's Lemma can be applied again. Now the centroid of the smaller triangle can be used as the initial guess in Newton's Method. If the answer again diverges, the new tricolored subtriangle, now one-eighty-first the original size, can be subdivided and the centroid of it found. It will always be the case that a tricolored subtriangle can be found which, according to Sperner's Lemma contains the Sperner point. Eventually, the centroid of a subtriangle will be close enough to the actual Sperner point so that Newton's Method will converge giving a good estimation of where all three paths are equal in length.

CONCLUSION

When looking at the traveling salesman problem, it is always possible to use Newton's Method, fueled with the centroid as an initial guess, to estimate the coordinate of the Sperner Point. When the number of cities is increased to four, it appears that at least one Sperner Point still exists. Future research may look into finding an initial point to use in Newton's Method so an estimate for the Sperner's Point of the trapezoid formed by the four cities will always converge.

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REFERENCE

[1] Bellino, Kathleen and Rekha Narasimhan. SUMSRI 1999 Journal. "On the Analytic Geometry of the Traveling Salesman Problem."