A Look into the Secret World of Primes

Thesis

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Abstract

Everyone is affected by primes on a daily basis, whether or not they are aware of it. The goal of this research is to explore the world of primes and extend my knowledge of them; then, through this knowledge to classify primes, to search for relationships between the classifications, and educate others about the uses of primes. This research resulted in programs that allow for the generation of various classifications and comparisons among the classifications. In the comparisons I found no patterns that would be useful in predicting the occurence of prime numbers.

Contents

Chapter 1

Introduction to Primes

1.1 Defining Prime

Upon entering grade school everyone learns basic counting: 1,2,3,4, and so on. At the time, no one explains that the numbers used for counting are called *natural numbers* (Bartle, 2). By the time students reach middle school negative numbers, fraction, and decimals have been introduced. They may or may not be told that these are respectively *negative integers*, *rational numbers*, and *real numbers* (Bartle, 2). Somewhere along the way students also learn to change numbers through addition, subtraction, multiplication, and division. For example, $2 + 2 = 4$ or $9 \div 3 = 3$.

Students also learn the important skills of finding factors and grouping numbers. Some examples, include: numbers divisible by 2 are called *even*, and integers greater than 1 whose only divisors are 1 and itself are called *prime* (Scheinerman, 1-3). When taught these skills students learn to count by twos and memorize the first few primes, 2,3,5,7,11, but that is the end of the story. Few students ever realize that the primes go on forever, and that they have haunted mathematicians for centuries.

One might be asking: how could a simple list of numbers haunt anyone? This question can be answered within minutes of investigating primes. Let's go back to the basic definition of what it means to be prime:

Definition 1.1.1 *an integer* $p > 1$ *is* prime *when its only divisors are trivial,* 1 *and* p. *Any number that is not prime is called* composite.

Despite this seemingly simple definition, primes hold great importance in the mathematical field. Prime numbers are the building blocks for all composite numbers. This paper will introduce some of the interesting and bewildering features of prime numbers. For now consider the statement: primes are not prime because humans say they are; they are prime simply by their existence (du Sautoy, 5-7). This sentence uses two definitions of the word prime to begin to describe the mystery and majesty of the prime numbers.

1.2 Determining Primeness

Now that the basic definition of a prime number is known, one can begin to expand the basic list of primes. To find more primes, one must establish ways to determine if a number is prime. Using only the definition requires checking to see if each integer less than that number is a divisor of that number. In other words, one must check to see if the number has any integers less than it that divide it evenly.

Example 1.2.1 *Determining if* 25 *is prime requires checking the integers* 2 *through* 24 *to see is any of them divide* 25 *evenly. In this case, one finds that* $25 \div 5 = 5$ *so* 25 *is not prime.*

This method is similar to the sieve of Eratosthenes and works nicely for small numbers (Derbyshire, 100-101). (See Appendix A for a detailed explanation of the sieve of Eratosthenes.) Both of these methods would require vast amounts of computations if trying to determine if a large number is prime. For such inquiries other methods are needed. More information is necessary before such methods can be found.

Theorem 1.2.1 *Any composite number can be written as a unique product of primes.*

> **Proof**: Let c be any composite number. Since c is not prime it has at least two divisors a and b such that both a and b are not equal to 1 or c, and a×b=c.
>
> > **Case 1**: a and b are both prime which means c is a product of two primes
> > **Case 2**: a and b are not prime so they can be written as a=s×t, such that both s and t are not equal to 1 or a, and b=u×v, such that both u and v are not equal to b.

This pattern will continue until $p_1 \times p_2 \times p_3 \times \ldots \times p_n = c$, where p_i are all prime numbers raised to some power.
Case 3: One of a and b is prime and the other is not. Let d be equal to which ever is prime of a and b and let e equal the one that is not prime. d can be expressed $d = \upsilon \times \omega$, such that both υ and ω are not equal to 1 or d. This pattern will continue until $d = p_1 \times p_2 \times p_3 \times \ldots \times p_n$. Since e is prime $c = d \times e = p_1 \times p_2 \times p_3 \times \ldots \times p_n \times e$, where p_i are all prime numbers raised to some power.

Now one must establish the uniqueness of such a factorization. The basic structure for this has been taken from *Prime Numbers A Computational Perspective* by Crandall and Pomerance. Here more detail is given. In order to prove uniqueness, some factual background is needed.

Lemma 1.2.1 ***Euclid's "first theorem"***: *The product of two integers is divisible by a prime p if and only if one of them is divisible by p (Crandall and Pomerance, 76).*

This is the basis for the proof of uniqueness. From the existence proof above it is known that c can be written in the form $c = p \times c'$ where p is a prime number and c' is the product of other primes. Assume by contradiction that c can also be written in the form $c = q \times c''$ where q is a prime distinct from p, and c'' is the product of other primes, not including p. The symbol $n|m$ denotes n divides m. Observe that since $q|c$, $q|p \times c'$. This implies that either $q|p$ or $q|c'$. Since q and p are both prime $q|p$ means that $q = p$, which is not true. This means that $q|c'$. If $q|c'$ this implies that $c' = q \times r_1 \times r_2 \times ... \times r_i$, where $r_1, r_2 \times, ..., r_i$ are primes. Hence, $c = p \times c' = p \times q \times r_1 \times r_2 \times ... \times r_i = q \times p \times r_1 \times r_2 \times ... \times r_i = q \times c''$. Since c'' does not include p a contradiction has been reached. Therefore c can be written as a unique product of prime number. Q.E.D.

Using this information about factoring, Pierre de Fermat derived a method for factoring large numbers. The formula he created takes $\triangle(x) = x^2 - n$, where n is the number you wish to factor and x is the next integer up from $\sqrt{n}$. If this number does not turn out to be a perfect square, find $\triangle(x+1) = (x+1)^2 - n = x^2 - n + 2x + 1 = \triangle(x) + 2x + 1$. Then, simply repeat the process several times until a perfect square (Ore, 56-57). Once reaching a perfect square you take the last $(x+a)$ and add and subtract the square root of the perfect square. The resulting numbers are the prime factors. It can be difficult to see how this works in theory so turn to the example 302303.

Example 1.2.2 *Begin with* $n = 302303$

$$\sqrt{n} = \sqrt{302303} \approx 549.8208799$$

so the next largest integer is 550*, giving* $x = 550$.
Substitute n *and* x *into* $\triangle(x) = x^2 - n$,

$$\triangle(550) = 550^2 - 302303 = 302500 - 302303 = 197$$

$$\sqrt{197} \approx 14.03566885,$$

since 197 *is not a perfect square the next part of the formula must be used.*

$$\triangle(550+1) = 197 + 2(556) + 1 = 1298,$$

$$\sqrt{1298} = 36.02776707,$$

so 1298 *is not a perfect square so the process must continue.*

$$\triangle(550+2) = 1298 + 2(550) + 3 = 2401,$$

$$\sqrt{2401} = 49$$

so 2401 *is a perfect square and we have now found the number needed to form the factors of* 302303.

$$(552+49)(552-49) = (601)(503) = 302303$$

This example appears easy, but with many numbers it takes hundreds of repetitions to find the prime factors using this method.

Another method for finding primes was developed by Euler. He found that the formula $x^2 + x + q$ would sometimes yield primes when fed numbers from 0 to $q - 2$ (du Sautoy, 45)

Example 1.2.3 *Using Euler's Formula,*

$$x^2 + x + q$$

Let $q = 15$, *so that means we feed the formula numbers from 0 to 13.*

$$0^2 + 0 + 15 = 15 = 3 \times 2,$$

$$1^2 + 1 + 15 = 17 = prime$$

$$2^2 + 2 + 15 = 21 = 3 \times 7,$$

$$3^2 + 3 + 15 = 27 = 3 \times 9,$$

$$4^2 + 4 + 15 = 31 = prime,$$

$$5^2 + 5 + 15 = 45 = 3 \times 15,$$

$$6^2 + 6 + 15 = 57 = 3 \times 19,$$

$$7^2 + 7 + 15 = 71 = prime,$$

$$8^2 + 8 + 15 = 87 = 3 \times 29,$$

$$9^2 + 9 + 15 = 105 = 3 \times 35,$$

$$10^2 + 10 + 15 = 125 = 5 \times 25,$$

$$11^2 + 11 + 15 = 147 = 3 \times 49,$$

$$12^2 + 12 + 15 = 171 = 3 \times 57,$$

$$13^2 + 13 + 15 = 197 = prime$$

Here 14 *numbers were tested and only four primes were found.*

This too seems very complicated and limiting. Perhaps there is another way to approach primes.

1.3 Guessing When and How Many Primes Occur

Now rather than trying to determine if a number is prime, consider ways to determine what the next prime will be. It has been established that the first few primes are 2,3,5,7, but how does one determine what the next prime is? With small numbers the methods defined in Section 1.2 will work, but problems occur with larger numbers. Perhaps the better path to take is to count the primes.

Theorem 1.3.1 *There are infinitely many primes.*

> **Proof**: Let $P(n)$ represent that there exists at least n primes. This will be shown by mathematical induction. First, it must be shown that $P(1)$, meaning that there is at least one prime, is true. By definition, 2 is a prime number, so there is at least one prime. Next, assume that P(k) is true for some natural number k, meaning that there are at least k primes. Finally, it must be proven that $P(k+1)$ is true, meaning that there are at least $k+1$ primes. Under the assumption that there are at least k distinct primes, the list $p_1, p_2, ..., p_k$ can be formed. Now, consider the number $\rho = p_1 \times p_2 \times, ..., \times p_k + 1$. Observe that for p_i, where $i = 1, 2, ..., k$, p_i cannot divide ρ. Clearly, ρ is either prime or not prime.
>
> > **Case 1**: ρ is prime. Another prime has been found and therefore it has been proven that there are at least $k + 1$ primes.
> >
> > **Case 2**: ρ is not prime. This implies that ρ has a prime factorization. Since p_i cannot divide ρ none of the prime factors of ρ equal p_i so there is at least one more prime than k.
>
> Q.E.D.

Now, counting the total number is out of the question. Two things that might be helpful to look at are the distance apart primes occur and the number of primes on a given interval. Let's begin by looking at the distance between each prime for the primes less than 100:

Prime	2		3		5		7		11		13		17
Distance		1		2		2		4		2		4	
Prime	17		19		23		29		31		37		41
Distance		2		4		6		2		6		4	
Prime	41		43		47		53		59		61		67
Distance		2		4		6		6		2		6	
Prime	67		71		73		79		83		89		97
Distance		4		2		6		4		6		8	

Table 1.3.1 ***Distance Between First 25 Primes*** *A pattern may appear to be developing, but upon a closer inspection no such pattern exists.*

Since there is not pattern in the distance between primes, another option is to take a closer look at the idea of counting the number of primes on a given interval. When looking at the primes less that 10 it is found there are four: 2,3,5,7. If the interval is extended to look at the primes less that 20, then 11,13,17,19 are added to the list. So far, it appears that there will be four new primes for every increase of ten. Increasing the interval by another ten to include the primes less than 30 adds only 23 and 29 to the list. Already, the pattern of four primes per ten integers has failed.

Continuing the approach of counting primes over intervals, let's try looking from another angle. From above, you can see that there are 4 primes less than ten, and 25 primes less than 100. Notice that $4 = 2^2$ and $25 = 5^2$. Perhaps, there is a pattern with the number of primes less than a power of ten being some number squared. Unfortunately, there are 168 primes less than 1000, and the pattern is lost because 168 is not a perfect square.

There seems to be no pattern to how often primes occur, which makes predicting the next one very difficult. Despite the seeming impossibility of this task, Gauss counted the primes on the prime tables available at his time and discovered:

Theorem 1.3.2 ***(known as The Prime-Number Theorem)****:*
$\pi(x)$ is the number of primes up to x. $\pi(x)$ behaves asymptotically like $\frac{x}{ln(x)}$ as their quotient approaches 1,

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{ln(x)}} = 1$$

(Ore, 77 and du Sautoy, 49).

Gauss claimed to have proof for this, but thought it was unnecessary to publish it. The Prime-Number Theorem was proven in 1896 independently by Jacques Hadamard and Charles de la Poussin (Derbyshire, 155). In simpler terms, this theorem describes that $\pi(x)$ acts like, or is a value close to, $\frac{x}{ln(x)}$. Notice here that this is only an approximation. Gauss himself was not satisfied with this approximation, so he found a more precise way to count primes by performing:

$$\frac{1}{ln(2)} + \frac{1}{ln(3)} + \ldots + \frac{1}{ln(x)} = \int_2^t \frac{dt}{ln(t)} \text{ now called Li}(x)$$

which led to the following theorem (Sautoy, 56).

Theorem 1.3.3 ***Prime Number Theorem(improved version)***:

$$\pi(x) \sim Li(x) \backsim \int_2^x \frac{dt}{ln(t)}$$

(Ore, 77 and Derbyshire, 116).

Although Gauss had found a closer approximation, it was still just an approximation. When dealing with approximations, one must wonder exactly how much error there is between the numbers the formula yields and the exact answer. Such ponderings led to the following theorem.

Theorem 1.3.4 ***Chebyshev's Second Result*** :
$\pi(x)$ cannot differ from $\frac{x}{ln(x)}$ by more than ten percent up or down (Derbyshire, 124)

This result is very important because it verifies that the $\pi(x)$ and Li(x) approximations of the number of primes up to a given number is fairly accurate. Although mathematicians have found no exact way to predict or produce primes, the methods that they have of counting primes has at most ten percent error.

The discussion of the distribution of prime numbers would not be complete without the mention of Mr. Bernhard Riemann and the Riemann Zeta

Function which has properties that seem to be very closely related to the distribution of primes (Ore, 78). Riemann was a German Protestant who was sick for most of his short life. By all accounts he was a shy man who kept to himself, but was very close to his family (Derbyshire, 28). Riemann had a love for the classical style of mathematics and based his efforts on ancient work (Sautoy, 61). The basic structure of his arguments depends greatly on the zeta function.

Definition 1.3.1 ***Zeta Function***:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} \ldots = \sum_{n=1}^{\infty} n^{-s}$$

(Ore, 78)

> **The Riemann Hypothesis**: All non-trivial zeros of the zeta function have real part one-half (Derbyshire, xi).

As stated above, the Zeta Function behaves much like the distribution of prime numbers, so finding the solutions to the zeta function would shed light onto the mystery of the primes. Although most math theorist believe the Riemann Hypothesis is true, no proof has been discovered. In fact, many mathematicians have assumed the truth of the Riemann Hypothesis in order to reach their personal goals (du Sautoy, 10). They must continue to wait for a proof for their own work to be validated. Such a proof would unlock many secrets of the mathematics community, and the disproof would cancel the work of thousands. The prospect of a proof of this hypothesis has led to private institutes offering millions of dollars for such a development, but no proof has emerged (Derbyshire, xi).

Chapter 2

Classification of Primes into Specialized Forms

Now that some general information about primes has been established, some special forms of primes will be investigated. To begin, let's look at pairs of primes that are related by how far apart the primes fall. The first few relationships consider the idea of adding a number to a prime number to generate another prime.

Theorem 2.0.5 *If an odd number is added to an odd number, the result is an even number.*

> **Proof**: An odd number is defined as an integer that can be written in the form, $x = 2n + 1$, where n is an integer. An even number is divisible 2. Let a and b be odd numbers such that $a = 2n + 1$ and $b = 2m + 1$. Then, $a + b = (2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1)$. 2 divides $a + b$, so $a + b$ is even. Q.E.D.

Observe that even numbers, with the exception of 2, cannot be prime because they are divisible by two. This leads to the conclusion that all primes must be odd. This information along with the theorem above shows that to form a new prime by adding a number to a prime, the number added must be even.

2.1 Twin Primes

The first even number is 2 so it would be logical to consider adding 2 to a prime, and checking if the resulting number is also prime.

Definition 2.1.1 *integers such that p, $p > 1$ and $p + 2$ are both prime are called* ***twin primes****.*

It seems that, with the exception of 2 and 3, the closest any two primes could be is twin primes.

Theorem 2.1.1 *No two prime numbers, excepting 2 and 3, can be closer than twin primes.*

> **Proof**:Let p not equal to 2 be a prime number. Since p is prime and not equal to 2, p must be odd. Hence, $p = 2n + 1$, where n is an integer. If we add one to p we get $p + 1 = 2n + 1 + 1 = 2n + 2$. By definition of even, $2n$ is even and an even number plus 2 is still even so $p + 1$ is even and therefore cannot be prime. Q.E.D.

Since there are infinitely many primes and the twin primes are the closest primes can be, it seems that there would be infinitely many twin primes.

Twin Prime Conjecture: *There are infinitely many primes such that p and $p + 2$ are both prime. (du Sautoy, 39)*

Unlike the infinity of primes there currently is no proof for this which is why it is called a conjecture and not a theorem. If proven this conjecture will be renamed to be a theorem.

I became very interested in the idea of two primes being so close so I wrote a program in Maple to generate primes over an interval and determine if they have a twin. (See Appendix B.2.) The following table shows the first twenty-five primes and if they have a twin.

prime(p)	p + 2	Twin
2	4	*no*
3	5	*yes*
5	7	*yes*
7	9	*no*
11	13	*yes*
13	15	*no*
17	19	*yes*
19	21	*no*
23	25	*no*
29	31	*yes*
31	33	*no*
37	39	*no*
41	43	*yes*
43	45	*no*
47	49	*no*
53	55	*no*
59	61	*yes*
61	63	*no*
67	69	*no*
71	73	*yes*
73	75	*no*
79	81	*no*
83	85	*no*
89	91	*no*
97	99	*no*

Table 2.1.1 ***Twins for the First 25 Primes***

Considering the set of twin primes, it can be observed that they seem to occur with less frequency the higher the number. This is logical, considering that the increasing primes thin in frequency.

2.2 Cousin Primes

Since one can only add even numbers to primes and have a possible prime the next even number to try adding is 4.

Definition 2.2.1 *an integers such that p, $p > 1$ and $p + 4$ are prime are called* ***cousin primes***

Notice here that the family of primes works similarly to our own families. The primes that are the closest are called twins. Now, the primes that are not as close are called cousin. Similar to the twins, I wrote a program in Maple to generate the primes and determine if they have cousins. (See Appendix B.3.) You can see the first 25 below.

prime(p)	p + 4	Cousin
2	6	*no*
3	7	*yes*
5	9	*no*
7	11	*yes*
11	15	*no*
13	17	*yes*
17	21	*no*
19	23	*yes*
23	27	*no*
29	33	*no*
31	35	*no*
37	41	*yes*
41	45	*no*
43	47	*yes*
47	51	*no*
53	57	*no*
59	63	*no*
61	65	*no*
67	71	*yes*
71	75	*no*
73	77	*no*
79	83	*yes*
83	87	*no*
89	93	*no*
97	101	*no*

Table 2.2.1 ***Cousins for the First 25 Primes***

Similar to the twin primes, notice that the larger the numbers get, the sparser the cousin primes become. Because of the randomness of the primes, I would propose that there are infinitely many cousin primes, but I have no way to prove such a statement.

2.3 Sexy Primes

Although these pairs of primes are quite intriguing, their name is not simply describing this quality. The name sexy come from the Latin word for 6, *sex*.

Definition 2.3.1 *integers such that* p*,* $p > 1$ *and* $p + 6$ *are prime are called* ***sexy primes***

Very similar to the twins and cousins, I wrote a program in Maple to generate the primes and then check to see if they have a sexy partner. (See Appendix B.4.) The table below shows the first 25.

prime(p)	p + 6	Sexy
2	8	*no*
3	9	*no*
5	11	*yes*
7	13	*yes*
11	17	*yes*
13	19	*yes*
17	23	*yes*
19	25	*no*
23	29	*yes*
29	35	*no*
31	37	*yes*
37	43	*yes*
41	47	*yes*
43	49	*no*
47	53	*yes*
53	59	*yes*
59	65	*no*
61	67	*yes*
67	73	*yes*
71	77	*no*
73	79	*yes*
79	85	*no*
83	89	*yes*
89	95	*no*
97	103	*yes*

Table 2.3.1 ***Sexies for the First 25 Primes***

Unlike with the twins and cousins, there is no obvious thinning in this list. Consider the next 25 primes to see if thinning occurs later.

prime(p)	p + 6	Sexy
101	107	*yes*
103	109	*yes*
107	113	*yes*
109	115	*no*
113	119	*no*
127	133	*no*
131	137	*yes*
137	143	*no*
139	145	*no*
149	155	*no*
151	157	*yes*
157	163	*yes*
163	169	*no*
167	173	*yes*
173	179	*yes*
179	185	*no*
181	187	*no*
191	197	*yes*
193	199	*yes*
197	203	*no*
199	205	*no*
211	217	*no*
223	229	*yes*
227	233	*yes*
229	235	*no*

Table 2.3.2 ***Sexies for the Primes from 100 to 230***

It can be seen that the sexy primes are beginning to thin, but it is obviously not as quick as the twins or the cousins. This gives evidence that there might be infinitely many sexy primes, but no proof currently exists.

At this point, there could have been section after section about adding even numbers to primes and checking to see if primes are generated, but for now I will leave this method and look at some other primes with special forms.

2.4 Germain Primes

Sophie Germain was a French woman who maintained correspondence with Gauss in late 17th and early 18th centuries, under the pretense that she was a man. Through this correspondence she found a special kind of prime that is now called a Germain prime (du Sautoy,193).

Definition 2.4.1 *an integer s such that $p > 1$ and $s = 2p+1$ are both prime is called a* ***Germain prime****.*

Below is a table I created using a program that I wrote in Maple to search for Germain primes. (See Appendix B.5.) This table is for all the primes less than 115.

prime	2(prime) + 1	is this prime
2	5	*no*
3	7	*yes*
5	11	*yes*
7	15	*no*
11	23	*yes*
13	27	*no*
17	35	*yes*
19	39	*no*
23	47	*yes*
29	59	*yes*
31	63	*no*
37	75	*no*
41	83	*yes*
43	87	*no*
47	95	*no*
53	107	*yes*
59	119	*no*
61	123	*no*
67	135	*no*
71	143	*no*
73	147	*no*
79	159	*no*
83	167	*yes*
89	179	*yes*
97	195	*no*
101	203	*no*
103	207	*no*
107	215	*no*
109	219	*no*
113	219	*no*

Table 2.4.1 ***Germain Primes Less Than 115***

Clearly, the table shows that the Germain primes thin as the primes increase. When I generated a larger number of the primes, they look to be thinning so greatly that one might be tempted to think they are a finite set, but no proof has been found.

2.5 Mersenne Primes

Marin Mersenne was a French monk who maintained correspondence with the French mathematician Pierre de Fermat about their mathematical ventures. Using this correspondence, Mersenne found information that resulted in a set of primes bearing his name.

Before exploring Mersenne's namesakes, let us look at Fermat's work that Mersenne used as a springboard. Fermat found that some number can be written in the form $F_t = 2^{2^t} + 1$. These are prime for the first few times, but then can be factored which was proven by Euler in 1739 (Ore, 74). Known as Fermat numbers, these numbers get large very quickly, and have been shown to have great significance. Gauss proved that this form is needed to create a perfect n-gons (du Sautoy, 39). A perfect n-gon is a closed shape with n sides, such that each side is of equal length and all the angles are equal.

Table 2.5.1 ***Example of a Perfect N-gon***

Fermat also found that prime numbers that have a remainder of 1 when divided by 4 can be written in as the sum of two squares (du Sautoy, 39) This result is the main thing that lead Mersenne to notice that many numbers could be written in the form $M_n = 2^n - 1$. These are now called Mersenne numbers (Ore, 71). Mersenne realized that if n was not prime, then M_n was not going to be prime. He also found that n being prime did not guarantee the primeness of M_n.

Definition 2.5.1 *When a prime can be written in the form $M_p = 2^p - 1$, where p is a prime number it is called a* ***Mersenne Prime****(Ore, 72).*

I wrote a program in Maple to find the Mersenne primes. (See Appendix B.6.) The table below shows the first 25 Mersenne numbers formed by primes and tells whether or not they are prime.

prime	$2^{\text{prime}} - 1$	is this prime
2	3	*yes*
3	7	*yes*
5	31	*yes*
7	127	*yes*
11	2047	*no*
13	8191	*yes*
17	131071	*yes*
19	524287	*yes*
23	8388607	*no*
29	536870911	*no*
31	2147483647	*yes*
37	137438953471	*no*
41	2199023255551	*no*
43	8796093022207	*no*
47	140737488355327	*no*
53	9007199254740991	*no*
59	576460752303423487	*no*
61	2305843009213693951	*yes*
67	147573952589676412927	*no*
71	2361183241434822606847	*no*
73	9444732965739290427391	*no*
79	604462909807314587353087	*no*
83	9671406556917033397649407	*no*
89	618970019642690137449562111	*yes*
97	158456325028528675187087900671	*no*

Table 2.5.2 ***Mersenne Primes for the First 25 Primes***

It can be see that the Mersenne numbers formed from primes get very large very quickly and that the ones that are prime get further and further apart. Like many other concepts about primes, it is believed that infinitely many Mersenne Primes exist, but there is no proof of such a notion.

Chapter 3

The Hunt for Relationships

I have discussed two different types of primes: primes grouped in pairs, and primes that meet some arithmetic formula. My next step was to investigate these two types and see if any relationship between them can be found.

3.1 Do Germain Primes Have Any Twin, Cousin, or Sexy Prime Partners?

To begin this search I used my program for generating Germain Primes and created additional code that would generate the twins, cousins, and sexies of these primes. (See Appendix B.7.) Below you see the table created by this program.

prime	2(prime) + 1	Twin	Cousin	Sexy
2	5	$3, 7$	N/A	11
3	7	5	$3, 11$	13
5	11	13	7	$5, 17$
11	23	N/A	19	$17, 29$
23	47	N/A	43	$41, 53$
29	59	61	N/A	53
41	83	N/A	79	89
53	107	109	103	$101, 113$
83	167	N/A	163	173
89	179	181	N/A	173
113	227	229	223	233
131	263	N/A	N/A	$257, 269$
173	347	349	N/A	353
179	359	N/A	N/A	353
191	383	N/A	379	389
233	467	N/A	463	461
239	479	N/A	N/A	N/A
251	503	N/A	499	509
281	563	N/A	N/A	$557, 569$
293	587	N/A	N/A	593
359	719	N/A	N/A	N/A
419	839	N/A	N/A	N/A
431	863	N/A	859	857
443	887	N/A	883	857
491	983	N/A	N/A	881

Table 3.1.1 ***Prime Partners for Germain Primes***

As can be see in the table, the Germain primes with twin partners are rare. Not shown on this table is that the number of Germain primes with twin partners continues to become more rare as the Germain primes increase. This is actually true for all the types of prime partners, even the sexy prime partners who may seem to not be thinning.

3.2 Do Mersenne Primes Have Any Twin, Cousin, or Sexy Prime Partners?

Now, I will follow the same path taken with Germain primes, but search for the twins, cousins, and sexies related to the Mersenne primes. Again, using a computer program I was able to compute these values. (See Appendix B.7.) The table below exhibits my results.

prime	$2^{\text{prime}} - 1$	Twin	Cousin	Sexy
2	3	5	7	*N/A*
3	7	5	3, 11	13
5	31	29	*N/A*	37
7	127	*N/A*	131	*N/A*
13	8191	*N/A*	*N/A*	*N/A*
17	131071	*N/A*	*N/A*	*N/A*
19	524287	*N/A*	*N/A*	*N/A*
31	2147483647	*N/A*	*N/A*	*N/A*
61	2305843009213693951	*N/A*	*N/A*	*N/A*
89	618970019642690137449562111	*N/A*	*N/A*	*N/A*
107	162259276829213363391578010288127	*N/A*	*N/A*	*N/A*
127	170141183460469231731687303715884105727	*N/A*	*N/A*	*N/A*

Table 3.2.1 ***Prime Partners for Mersenne Primes***

The occurrence of Mersenne primes having any partners continues to be extremely rare with larger numbers. I would attribute this to how large the Mersenne primes get so quickly, but I have no proof of this being the cause for the rareness.

3.3 Results of My Search

As seen in the previous two sections, my search for relationships between being a Germain or Mersenne primes, and having a twin, cousin, or sexy partner has not revealed any divine answer to the mystery of the primes. Despite the fact that I have yielded no extreme revelation, my efforts have not been completely unfruitful. I have gained vast amounts of knowledge about the different types of primes and verified that there is not a relationship between the types. I have also developed many useful programs which you will find the code for in the Appendix B. Most importantly I have gained the experience of writing programs in Maple and of writing in a TEX editor. My results are of no great importance to the mathematical community, but they have laid the ground work for my future endeavors.

Chapter 4

Uses of Prime Number

Although my search for a pattern in the prime numbers was not as successful as I would have hoped, primes numbers are part of our every day lives. There are many places that we see prime numbers if we just look around. For example, there are 3 colors in the United States flag, 7 days in a week, and 31 days in many of our months. The uses of primes stretch far beyond random occurrences and theoretical mathematics. Scientists have found uses for primes in the fields of physics, engineering, chemistry and biology (Crandall and Pomerance, 388).

A place where primes appear in biology is in the life cycles of periodical cicadas. Their life cycles are a prime number of years long, mostly 17 and 13 years. These life cycles of prime number of years are beneficial to the cicadas because of the indivisibility of prime numbers. What this means in the long term affects is that it takes several life cycles for the cicadas and its predator's life cycles to align (Crandell and Pomerance, 189-9).

A scientific application of prime numbers appears in the business world. Computer scientist have found ways to use prime numbers to create a more secure business environment. If you have ever purchased anything using a credit or off of the internet, then you have experienced this use of primes. In the 1970's Ron Rivest, Adi Shamir, and Leonard Adleman developed a way to use prime numbers to encode credit card numbers and personal information. Now in our modern age of e-business, their system, called RSA, is still using hundred digit prime numbers to keep internet shoppers safe. This system uses the products of large primes to form numbers that have only two prime factors to encode informations. The security is in the difficulty to factor such large numbers (du Sautoy, 11-12).

Outside of the scientific realm, prime numbers have influenced more aesthetic areas. When you think of prime numbers, unless you are a musician, you probably do not think of counting out rhythms music or intervals between the notes. Primes came to use for Olivier Messiaen while he was imprisoned in the late 1930's and composing *Quartet for the End of Time*. In this piece he used a variety of rhythms that created a very different sound (Riley, 193). Many of the rhythms and note intervals are based on prime numbers, such as sets of 17 notes. Other musicians have used larger primes such as 37, 41, 43, 47, and 53 (Crandall and Pomerance, 394).

Finally, a simple application of primes that anyone can perform is a card trick. The idea to the card trick is to make the audience believe that you can produce a chosen card which achieved by shuffling the deck in a systematic way. You begin by taking a prime number of cards from a regular deck of playing cards. This trick is a great use for that deck of cards that is mission a card or two. The first step is to shuffle the cards. Then, you look at the bottom card, show it to the audience, and return it to the bottom of the deck turned so that the card faces the opposite direction of the rest of the cards.

Next, you ask someone to pick a number between 1 and the number of cards less 1. For example, if you are working with 13 cards, you ask them to pick a number between 1 and 12. One way to accomplish the card trick is to perform some quick mental math to see how many times you must add the number they chose to itself to get a multiple of the number of cards in your deck. Returning to the example of a deck of 13 cards, if they chose 4 cards: $4+4+4+4+4+4+4+4+4+4+4+4+4 = 52$ so it would take 13 additions to return to the original order of the deck. However, since you are working with a prime number of cards, it will always take the same number of revolutions as the number of cards in the deck, so the mental math is not necessary.

Next, you ask someone to remove the number chosen less 1, one at a time and place them at the bottom of the deck. If 4 is the chosen number they take 3 card one at a time and place them at the bottom of the deck. Then, ask them to turn the next card face up and place it at the bottom of the deck. You then repeat this process the number of times that you have cards in your deck. Then you can state that the card shown at the beginning of the trick will be the only card left facing the opposite direction (Eastaway & Wyndham, 147).

You have seen that the prime numbers are a random set that seem to fit no molds and refuse to follow any patterns. However, it is this randomness that makes primes so useful and inspiring. Man and nature both use the primes to create security and form beauty. Perhaps you are now inspired to dive into the secret world of primes.

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Appendix A

Sieve of Eratosthenes

Begin with a list of numbers. It is known that 1 is not prime. The first prime is 2 so any number that has two as a factor cannot be prime. 2 is left alone, but all the numbers that are multiples of 2 are removed from the list.

	~~2~~	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~
51	~~52~~	53	~~54~~	55	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	63	~~64~~	65	~~66~~	67	~~68~~	69	~~70~~
71	~~72~~	73	~~74~~	75	~~76~~	77	~~78~~	79	~~80~~
81	~~82~~	83	~~84~~	85	~~86~~	87	~~88~~	89	~~90~~
91	~~92~~	93	~~94~~	95	~~96~~	97	~~98~~	99	~~100~~

After 2, 3 is the next primes. Like with 2, 3 is left alone, but all the multiples of 3 are removed from the list.

	2	3		5		7		~~9~~	
11		13		~~15~~		17		19	
~~21~~		23		25		~~27~~		29	
31		~~33~~		35		37		~~39~~	
41		43		~~45~~		47		49	
~~51~~		53		55		~~57~~		59	
61		~~63~~		65		67		~~69~~	
71		73		~~75~~		77		79	
~~81~~		83		85		~~87~~		89	
91		~~93~~		95		97		~~99~~	

As before, 5 is left alone, but all the multiples of 5 are removed.

	2	3		5		7			
11		13				17		19	
		23		~~25~~				29	
31				~~35~~		37			
41		43				47		49	
		53		~~55~~				59	
61				~~65~~		67			
71		73				77		79	
		83		~~85~~				89	
91				~~95~~		97			

Continue as above with 7.

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47		~~49~~	
		53						59	
61						67			
71		73				~~77~~		79	
		83						89	
~~91~~						97			

Then, one can proceed in this same manner with each number reached that is left on the list. In this case 7 is the last number that has multiples left on the list. Here the sieve of Eratosthenes was used to find the primes less than 100. The same process could be used over any interval.

Appendix B

Maple Programs

B.1 Prime Generator

B.2 Twin Primes

B.3 Cousin Primes

B.4 Sexy Primes

B.5 Germain Primes

B.6 Mercene Primes

B.7 Germain Prime Pairs

B.8 Mersenne Prime Pairs