Lecture 3.2

Projectile Motion

Last time we were talking about two-dimensional motion and introduced all important characteristics of this motion, such as position, displacement, velocity and acceleration. Now let us see how all these things are interrelated in the special case of constant acceleration. We already know that in this case equations of motion will have the form

\[
\begin{align*}
\ddot{a} &= \text{const}, \\
\dot{a}_x &= \text{const}, \\
\dot{a}_y &= \text{const}. \\
\ddot{v} &= \ddot{v}_0 + \dot{a}t, \\
v_x &= v_{0x} + \dot{a}_x t, \\
v_y &= v_{0y} + \dot{a}_y t. \\
\ddot{r} &= \ddot{r}_0 + \ddot{v}_0 t + \frac{1}{2} \ddot{a} t^2, \\
x &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2, \\
y &= y_0 + v_{0y} t + \frac{1}{2} a_y t^2.
\end{align*}
\] (3.2.1, 3.2.2, 3.2.3)

As we have already discussed, all the objects have the same absolute value of acceleration, \( g = 9.8 \text{m/s}^2 \), when they are falling down near the earth’s surface and this acceleration is pointing downwards to the earth's surface. To simplify the problem we shall ignore air resistance, which is quite accurate approximation as far as speed of the object is not too large and there is no wind. We shall only consider the problem in two dimensions. So, we will only study the object which moves in a vertical plane. We choose the \( x \)-axis to be horizontal axis in the same plane as the object moves and axis \( y \) to be a vertical axis in the same plane with positive direction going upwards. The object has an original velocity, \( \ddot{v}_0 \), in the same plane (see the picture). Such an object is called projectile and its motion a projectile motion.
We can resolve the original velocity into two components which is

\[ \vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}, \]

\[ v_{0x} = v_0 \cos \theta_0, \]

\[ v_{0y} = v_0 \sin \theta_0. \]  

(3.2.4)

Here, \( \theta_0 \), is the angle between the direction of axis \( x \) and the original velocity \( \vec{v}_0 \). We can then resolve the acceleration vector, but it is even easier. The only component that acceleration has is a vertical component

\[ a_x = 0, \]

\[ a_y = -g, \]  

(3.2.5)

Since gravitational acceleration is constant we can consider this problem using the set of equations 3.2.1-3.2.3.

It can be reduced to two one-dimensional problems, because for a projectile \textit{vertical motion and horizontal motion are completely independent}. This fact can be confirmed experimentally by watching two balls, one of which is falling straight down, another is shot horizontally by a spring. Their motion in vertical direction will be exactly the same. They will cover the same distances during the same time intervals and it takes the same time for both of them to reach the ground.
For horizontal motion we have

\[ a_x = 0, \]
\[ v_x = v_{0x} = v_0 \cos \theta_0 = \text{const}, \]
\[ x = x_0 + v_{0x} t = x_0 + v_0 \cos \theta_0 t. \]

(3.2.6)

For vertical motion, the similar set of equations gives

\[ a_y = -g = \text{const}, \]
\[ v_y = v_{0y} + a_y t = v_0 \sin \theta_0 - gt, \]
\[ y = y_0 + v_{0y} t + \frac{a_y t^2}{2} = y_0 + v_0 \sin \theta_0 t - \frac{gt^2}{2}. \]

(3.2.7)

The last set of equations illustrates behavior of the vertical component of velocity depicted in my picture. At first it is equal \( v_{0y} \) then it gets smaller due to deceleration of the object by gravitational field. At the highest point of the trajectory the vertical component of velocity becomes zero, so it only moves horizontally, then it is accelerated in the downward direction by the same gravitational field. Finally it reaches the ground with the same speed as it had originally, and it makes the same angle below the horizon as the original angle above the horizon was.

The trajectory may look like a complicated curve. However, it can be shown by eliminating time from equations 3.2.6 and 3.2.7, that it is just parabola

\[ y = \tan \theta_0 x - \frac{gx^2}{2(v_0 \cos \theta_0)^2}, \]

(3.2.8)

(for simplicity, we set \( x_0 = 0, \ y_0 = 0 \)).

**Exercise:** Prove equation 3.2.8.

**Exercise:** How equation 3.2.8 will look like if \( x_0 \neq 0 \) and \( y_0 \neq 0 \)?

In the picture you can also see the horizontal distance, \( R \), which projectile has traveled before it returns to its initial launching level. Let us find this distance, called the horizontal range of the projectile.

For the coordinates at the final point one has

\[ x - x_0 = R, \]
\[ y - y_0 = 0. \]

With account of equations 3.2.6, 3.2.7 it becomes
\[ R = v_0 \cos \theta_0 t, \]
\[ 0 = v_0 \sin \theta_0 t - \frac{gt^2}{2}. \]

Eliminating time from these two equations one has
\[ R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0 = \frac{v_0^2}{g} \sin 2 \theta_0. \]  \hspace{1cm} (3.2.9)

**Exercise:** Prove equation 3.2.9, by eliminating time from the previous equation.

This last equation can help us to answer the interesting question: At which angle one needs to shoot from the gun, in order to get the largest horizontal range \( R \)? If \( R \) has the largest value for the given original speed of projectile, this means that sine of the angle in equation 3.2.9, should reach its maximum value, which is 1, so we have
\[ \sin 2 \theta_0 = 1, \]
\[ 2 \theta_0 = 90^\circ, \]
\[ \theta_0 = 45^\circ. \]

This means that horizontal range will reach its maximum, if you launch a projectile at 45 degrees angle and it is equal \( R_{\text{max}} = \frac{v_0^2}{g} \sin 90^\circ = \frac{v_0^2}{g} \).

**Example 3.2.1.** At what angle \( \theta_0 \) relative to the horizon one has to shoot a projectile in order to have the projectile's height at the highest point equal projectile's horizontal range?

We just proved that projectile's horizontal range is \( R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0 \). Now let us see what projectile's height is. Since at the highest point of the trajectory the vertical component of velocity is zero, we have
\[ v_y^2 = v_{y0}^2 + 2a_y (y - y_0), \]
where \( v_y = 0, \ v_{y0} = v_0 \sin \theta_0, \ a_y = -g \) and \( y - y_0 = H \), so
\[ 0 = v_0^2 (\sin \theta_0)^2 - 2gH, \]
\[ 2gH = v_0^2 (\sin \theta_0)^2, \]
\[ H = \frac{v_0^2 (\sin \theta_0)^2}{2g}. \]
In this problem we have $H=R$, so

\[
\frac{v_0^2 (\sin \theta_0)^2}{2g} = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0,
\]

\[
\sin \frac{\theta_0}{2} = 2 \cos \theta_0,
\]

\[
\frac{\sin \theta_0}{\cos \theta_0} = 4,
\]

\[
\tan \theta_0 = 4,
\]

\[
\theta_0 = \tan^{-1}(4) = 76^\circ.
\]