Lecture 4.2
Forces, Newton's Second Law

Last time we have started our discussion of Newtonian Mechanics and formulated Newton’s laws. Today we shall closely look at the statement of the second law and consider some practical examples.

1. Common Forces

Newton's second law states that: *The net force acting on a body is equal to the product of the body’s mass and the acceleration produced by this force* or

\[ \vec{F}_{\text{net}} = m\vec{a} \, . \]  

(4.2.1)

In order to be able to apply Newton’s second law we have to consider some common examples of forces which you can observe in the every-day life. The most well-known example is *gravitational force*. This force acts between any two objects in the universe. However, we usually refer to it as gravitational force of the earth. The reason is that gravitational force is proportional to the masses of the two objects involved (We shall study this in details later), but coefficient of proportionality is too small. This means that the mass of at least one of the objects have to be huge in order for us to be able to observe this force. This is why everybody can “feel” gravitational force of the earth, but we can not detect this force caused by the other objects, since their masses are very small compared to mass of the earth.

It is experimental fact that the earth provides the same acceleration (in the absences of air resistance) to all the objects near its surface. We have already discussed that when studying kinematics. The only way for this to be possible is if gravitational force is proportional to the mass of the object it acts on, so

\[ m\vec{a} = \vec{F}_{\text{net}} = \vec{F}_{\text{gravity}} = m_g \vec{g} \, , \]

resulting in \( \vec{a} = \vec{g} \) for any object. Note that we use letter, \( m_g \), meaning gravitational mass, when representing gravitational force in contrast with just \( m \), known as inertial mass in the Newton's second law. This is because we do not have any theoretical evidence that these are the same masses, it is just experimental result, known with very high accuracy. So in future we will use the same letter \( m \) for both gravitational and inertial masses. The gravitational force stays of the same magnitude even if the object is not falling, which is

\[ F_g = mg \]  

(4.2.2)
even if it is not the only force acting on the body, but just one of the many forces.

We shall define weight of the body as the magnitude of the force, which prevent this body from falling freely as measured by someone on the ground. For instance, if a person standing on the ground holds a ball in his/her hand, he/she has to provide the force on the ball to prevent it from falling. Since the ball does not move and has zero acceleration, the force provided by the person has to be equal in magnitude, \( W \), but opposite in the direction to the gravitational force acting on the ball, so

\[
ma = W - F_g, \\
0 = W - mg, \\
W = mg.
\]

This means that the weight \( W \) of body has the same magnitude as gravitational force acting on it. However, this statement is only true in inertial reference frame. If you try to perform weighting of the body in the elevator moving with some nonzero acceleration, you can come up with some very strange results. For instance, if the elevator is falling down, the body will be weightless at all. Another aspect which needs to be mentioned is that weight and mass are not the same things. Even though weight is proportional to mass it can be quite different if you perform weighting of the same object with a very same mass, let us say on the Moon instead of the Earth.

Another example of force, which we can observe in every-day life, is normal force. The normal force caused by the compressed surfaces and acts perpendicular to those surfaces. This is why it is called the normal force. For instance, if you are standing on the floor then there is gravitational force acting on you and you act with the same force equal to your weight on this floor. As a result of this action, the floor is compressed and pushes back with its normal force. The same type of behavior takes place for any surface. For some soft enough surfaces as mattress, you can even see how it is compressed. For other rigid surfaces, you can not see the effect of compression, since it is too small, but the reason for the normal force is still the same.

If we consider a block placed on the smooth surface we can write the Newton's second law as

\[
\tilde{m}{\ddot{a}} = \tilde{N} + m\tilde{g}, \\
\tilde{N} = m\tilde{a} - m\tilde{g}.
\]

This last equation gives us the magnitude and direction of the normal force. In fact, we already know that direction of the normal force is always perpendicular to the surface, so
we can rather say that these equations define direction of acceleration $\vec{a}$. In the case if the block is not moving and the surface is horizontal the second equation gives us the magnitude of the normal force as

$$N = mg.$$  \hfill (4.2.5)

But this is only true if $a=0$ and for horizontal surfaces.

Next, we shall consider the force of tension $\vec{T}$. This force has the nature similar to the nature of the normal force. It occurs inside of the different types of cords or strings attached to the body and acts in the direction from the body along the cord. The cord is often said to be massless and unstretchable. This means that the force of tension has the same magnitude at both ends of the cord. Since the cord is usually connecting two bodies, this means that the same (in magnitude) tension acts on each of those bodies and directions of these tensions are always along the cord.

Another example of a typical force is the force produced by a compressed or stretched spring. Let us consider a box on horizontal floor attached to the originally relaxed (neither compressed no stretched) horizontal spring. Another end of this spring is attached to the wall. If a person starts pulling or pushing the box, he/she will compress or extend this spring. Experiment shows that in such a case the spring will respond by acting on the box with restoring force, which tries to bring the spring back to its original non-deformed state. This force is always directed in the direction opposite to the force applied by the person. It is known from experiment, that this restoring force is proportional to the spring's displacement from its equilibrium. This is true, at least, for small displacements. Mathematically we can write this statement as

$$\vec{F} = -kd.$$  \hfill (4.2.6)

This last equation is known as the Hooke's law after Robert Hooke, an English scientist of the late 1600s. The minus sign in this equation shows that direction of force is opposite to the spring's displacement. The proportionality coefficient $k$ stands for the so-called spring constant, which measures the stiffness of the spring and has its own value for any particular spring. The SI unit for this constant is $N/m$. We can choose the direction of axis $x$ along the displacement vector with the origin at the spring’s equilibrium position, so the Hooke's law becomes

$$F = -kx.$$  

This force gives us an example of the variable force, which changes with change of $x$. 
2. Simple Applications

We are now ready to apply Newton's second law in practice. According to the this law

$$\sum \vec{F} = m\ddot{a}. \quad (4.2.7)$$

The symbol $\sum$ stands for summation, since the law includes not just some force, but the net force acting on the body, which is a vector sum of all the forces. Equation 4.2.7 can also be rewritten in terms of vector components as

$$\sum F_x = ma_x, \quad (4.2.8)$$
$$\sum F_y = ma_y,$n
$$\sum F_z = ma_z.$n

In the most part of the cases, we will have problems not in three, but in two dimensions, which reduces the number of equations to two. This means that if you have only one body in the problem, it cannot contain more than 3 (in 3 dimensions) or more than 2 (in 2 dimensions) unknowns. However, we will consider some problems with not just one but several interacting bodies. In this case the number of unknowns can be higher, but in any case not more than the number of equations.

In order to be able to write down the set of equations set 4.2.8, first you have to draw a free body diagram and choose the appropriate coordinate system.

Follow these simple steps for drawing diagrams:

1. Isolate the body in the question.
2. Identify all the forces acting on the body and draw them in your diagram representing as vector-arrows in appropriate scale.
3. Choose the coordinate system in such a way that your vectors have the simplest components. Identify you choice of coordinate system in the picture.
4. Add the forces vectorially in the picture and mathematically using the set of equations 4.2.8.

After you have finished drawing free body diagram, you have to proceed with solution of the set 4.2.8, obtaining the unknown variables.

Now, when we have studied so many different forces, we can use this knowledge to solve various problems.
The picture above shows a block of mass, $M$, suspended on three cords. The cords are connected in knot $K$. The first cord makes angle $\alpha$, and the second cord makes angle $\beta$ with the ceiling. These cords and the knot are massless. What are the tension forces in each of the cords?

Each of the unknown tensions $T_1$, $T_2$ and $T_3$ is shown in the picture. Tension $T_3$ is shown twice, because even though it has the same magnitude but it is applied to the different objects. It acts on the knob (the red vector in the picture) at one side of the cord and it acts on the block (the blue vector in the picture) at the other side of the cord. First, we shall consider the block to find the unknown tension $T_3$. There are two forces acting on the block: tension $\vec{T}_3$ and gravitational force $\vec{F}_g = Mg$. The block does not move, so it has zero acceleration and Newton's second law for this block reads

$$\vec{T}_3 + Mg = 0,$$
$$T_3 - Mg = 0.$$  \hfill (4.2.9)

The first of these equations is in vector notation, while the second represents components of vectors on axis $y$ (see the picture). We have chosen the $y$-axis to be in the vertical
upward direction and the $x$-axis in horizontal direction from the left to the right. So, the answer for the magnitude of tension $T_3$ is very simple, it is

$$T_3 = Mg. \quad (4.2.9)$$

This tension is equal to the weight of the block. Now let us find two other unknown tensions. To do so we shall use Newton's second law for the knot. There are three forces acting on it. These are the three tensions. The knot is not moving, so it has zero acceleration and Newton's second law gives

$$T_1 + T_2 + T_3 = 0. \quad (4.2.10)$$

To solve this vector equation for two unknowns, we have to resolve it into components for $x$ and $y$-axes. One can see from the picture that

$$T_{1x} + T_{2x} = 0,$$
$$T_{1y} + T_{2y} + T_{3y} = 0, \quad (4.2.11)$$

or

$$T_2 \cos \beta - T_1 \cos \alpha = 0,$$
$$T_1 \sin \alpha + T_2 \sin \beta - T_3 = 0. \quad (4.2.12)$$

The first of these equations gives

$$T_2 = T_1 \frac{\cos \alpha}{\cos \beta}. \quad (4.2.13)$$

Substituting this into the second equation we have

$$T_1 \sin \alpha + T_1 \frac{\cos \alpha}{\cos \beta} \sin \beta - T_3 = 0,$$
$$T_1(\sin \alpha + \cos \alpha \tan \beta) = T_3 = Mg,$$\quad (4.2.14)
$$T_1 = \frac{Mg}{\sin \alpha + \cos \alpha \tan \beta}.$$  

This gives the answer for the tension in the first cord. Similar, the tension in the second can be determined as

$$T_2 = \frac{Mg}{\sin \alpha + \cos \alpha \tan \beta} \frac{\cos \alpha}{\cos \beta} = \frac{Mg}{\tan \alpha \cos \beta + \sin \beta}. \quad (4.2.15)$$

The answer for the second tension has completely symmetric form compared to the answer for the first tension.

**Exercise:** Consider several cases of this problem, when angles $\alpha$, $\beta$ or both of them equal zero degrees or ninety degrees. What is the physical significance of these situations?
In the problem which we have just considered, acceleration of the bodies was zero, so this was the case of equilibrium. Let us now consider a different example, where the object is moving, and see how this affects the problem.

**Example 4.2.2.** A person of mass $m$ is standing on the scale in the elevator. What is the scale reading if (a) the elevator is not moving, (b) the elevator is moving with constant velocity up or down (c) the elevator moves down accelerating or the elevator moves up decelerating, (d) the elevator moves up accelerating or the elevator moves down decelerating. For parts (c) and (d) assume that the absolute value of acceleration is $a$.

Let us first solve this problem in general. The reading of the scale is always equal to the force, which acts on the scale from the person. According to the Newton's third law this force is equal in magnitude to the force acting from the scale on the person, which is normal force $\vec{N}$. So the scale always shows the magnitude of force $\vec{N}$ and this is the force we have to find.

In this example all the forces are acting along the vertical direction, so we only need one axis $y$, which we can choose to be upwards. The only two forces acting on the person are the gravitational force $\vec{F}_g = mg$ and the normal force $\vec{N}$. We shall write Newton's second law for the person in the elevator as

$$m\ddot{a} = mg + \vec{N}, \quad \text{(4.2.16)}$$

so the unknown normal force will be

$$\vec{N} = m\ddot{a} - mg. \quad \text{(4.2.17)}$$

This is the general solution of the problem. Now let us see how it works for different situations.

We can consider cases (a) and (b) together. This is just a consequence of the Newton's first law, which states that there is no difference between the motion with constant velocity and zero velocity. In both these cases $\ddot{a} = 0$, so equation 4.2.17 becomes

$$\vec{N} = -mg. \quad \text{(4.2.18)}$$

This means that the scale will show the real weight of the person which is equal to gravitational force ($\vec{N} = mg$).

In case (c) we have acceleration directed downward, regardless of whether or not the elevator is moving down accelerating or up decelerating. Equation 4.2.17 in terms of vector components gives
\[ N = -ma - (-mg), \]
\[ N = mg - ma, \]
\[ N = m(g - a). \]  

(4.2.19)

This means that the scale's reading will be less than the real weight of the person. You can even feel it sometimes when moving in an elevator, especially when in the skyscraper, where elevator should have the large value of acceleration in order to be able to develop the large speed to cover the large building height in a short time. In the case if \( a = g \), \( N=0 \) and the person will fly weightless.

In case (d) acceleration is directed in the upward direction, so the equation becomes
\[ N = ma - (-mg), \]
\[ N = ma + mg, \]
\[ N = m(a + g). \]  

(4.2.20)

The scale shows the reading, which is higher than the real person's weight.

### 3. Forces of Friction and Air Resistance

Until this moment we were always ignoring friction and air resistance in all of the problems we have considered. Now it is time to learn more about frictional force. This is because even if this force is small it is still unavoidable in everyday life. Moreover, this force is, in fact, the one of the great importance.

*Think why friction is so important.*

Indeed it would not be possible for us to drive a car if there is no friction between the car's tires and the road. It becomes dangerous even if this friction is just reduced during the rainy day or on the icy road. We would not be even able to walk without friction, like if you were trying to walk on ice.

We shall consider friction between dry solid surfaces either stationary or moving relative each other at low speed. Let us look at some general properties of frictional force for an object placed on or moving along the surface. We know that a hockey puck moving along the horizontal surface will finally stop unless an external force is applied to keep it moving. This means that there is decelerating force acting along the surface in the direction opposite to the puck's velocity. This decelerating force is the force of friction. So, we can conclude that frictional force always acts along the surface in the direction opposite to the object's velocity. If we keep pushing the puck in such a way that it does not stop, we can choose the pushing force so that the puck is moving with constant speed.
If it moves with constant speed then there is no acceleration. But we were applying force all the time. If this force does not provide acceleration, this means that horizontal component of the applied force is balanced by the force of friction.

*Based on this, think how one can measure, the force of friction. You will need this during the lab next week.*

Frictional force is not always the same even for the chosen surface. Indeed, if we push a heavy box placed on a horizontal floor and it is not moving then the horizontal component of the pushing force is equal to the frictional force. If we push harder it still may not be moving. This means that the harder we push the stronger frictional force will become. But there is a limit; if we push real hard the box will finally move.

The example, which we just considered, illustrates two types of frictional forces. In the first case (when the object was not moving) the frictional force acting on the box is the static frictional force $F_{fs}$ and it is equal to the pushing or pulling force in horizontal direction. But this force has a limit. After this limit is reached the object starts moving and there is kinetic frictional force $F_{fk}$ acting on it during the motion. Usually the magnitude of kinetic frictional force is less that the magnitude of the maximum static frictional force

$$F_{fk} < F_{fs,\text{max}}. \quad (4.2.21)$$

You can also notice that it is much harder to move a heavy box compared to the empty box. This means that frictional force depends not only on the surface itself, but also on the mass of the body. In fact friction has to do not with the mass but with how hard the body is pushing on the surface. If it pushes harder it makes more contacts with the surface which increases frictional force. So the larger the normal force the larger the friction will be. It is experimental fact that this relation between the frictional force and the normal force is linear relationship which means

$$F_{fs,\text{max}} = \mu_s N. \quad (4.2.22)$$

Proportionality coefficient $\mu_s$ in this relation is known as the coefficient of static friction. In a same way for kinetic friction one has

$$F_{fk} = \mu_k N, \quad (4.2.23)$$
where $\mu_k$ is the coefficient of kinetic friction. Coefficients of friction depend on the surface properties of both bodies in contact. As you can see from its definition, it is dimensionless, so it does not have any units.

*Think how one can measure coefficient of friction. You will do it in the lab next week.*

We shall now look at what happens to a body, when it is moving inside of gas or liquid (generally known as motion in fluid). In this case the body will experience a *drag force* $D$. Once again this drag force is directed opposite to the direction of the body’s velocity, but in contrast to the dry friction, it also depends on this velocity. We shall only look at one example, where a smooth body moves fast enough in the air, so that the air becomes turbulent behind the body. It is experimental result that in such a case, the absolute value of the drag force can be found as

$$D = \frac{1}{2} C \rho A v^2,$$ \hspace{1cm} (4.2.24)

Where $C$ is known as the drag coefficient, which can be considered approximately constant for a given body, $\rho$ is the air’s density (mass per unit of volume), $A$ is the area of the body’s cross section taken perpendicular to the body’s velocity and $v$ is the body’s speed relative to the air.

We can now account for the air resistance force during the free fall of the object. If the object has the appropriate shape and the speed of the object is large enough, equation 4.2.24 is applicable and Newton’s second law for the object falling straight down becomes

$$m \ddot{a} = D + mg,$$

$$m \ddot{a} = \frac{1}{2} C \rho A v^2 - mg$$

As velocity of the body grows at first, the drag force becomes higher and acceleration decreases. At some point the drag force will be so large that acceleration will become zero. That happens when velocity reaches its terminal value $v_t$. We can find the terminal velocity from the last equation, by setting acceleration to zero, which gives

$$0 = \frac{1}{2} C \rho A v_t^2 - mg,$$

$$v_t = \sqrt{\frac{2mg}{C \rho A} \hspace{1cm} (4.2.25)}$$