Lecture 6.1

Work and Energy

During previous lectures we have considered many examples, which can be solved using Newtonian approach, in particular, Newton's second law. However, this is not always the most efficient way to deal with problems in physics. Indeed, let us consider an example where a car is moving down the hill of the known original height. If this hill can be considered as a perfect inclined plane with the known angle of elevation, we can easily solve the problem and find the car's speed at the bottom of the hill. We have already done this several times. On the other hand only a few real hills could be considered as perfect inclines. In fact, the hills usually have very unpredictable shape with ups and downs. So, what do we do in such a case? Shall we choose the appropriate coordinate system for every small part of this hill and solve the problem numerous number of times? We can do that, but how much time will it take? One of the reasons for this difficulty is that Newton's equations are vector equations, so they are very sensitive to the choice of coordinate system. If we could find a way to describe this motion by means of some scalar quantity which, of course, is a function of the body's position, but, at least, it is not changing rapidly with change of direction for the coordinate axes, then we could find a shorter and easier method to solve this problem. This method does exist and it is known as work-energy approach.

It is clear that work and energy must be scalars. Work is associated with the process of energy change or transfer, while energy itself is associated with the state of the system. Later we shall see that different types of energy may exist in nature. Kinetic energy is associated with moving objects. At the same time potential energy is not necessarily attributed to the moving object, but to any object which is potentially capable of performing work. These two types of energy are examples of mechanical energy. There are also other types of energy, such as thermal energy, chemical energy, electromagnetic energy and many more.

Let us start from definition of work.

1. Work

Let us consider an object under the influence of some force. It can be a car going down the hill under the influence of gravitational force or it can be a box on a floor which
is pulled or pushed along this floor by a person. In any case the state of the object will be changing under the influence of force. The object is either moving faster or slowing down depending on the force acting on it. If we associate a scalar quantity, which we shall call kinetic energy with the moving object, it will be changing with changes of the object's speed. This means that the force acting on the object performs work which goes into change of the object's kinetic energy. It is said that work, $W$, is done on the object by the force.

Before defining work mathematically, we shall go back to our discussion of vectors and recall the definition of the scalar product of the two vectors. According to this definition, the scalar product of two vectors $\vec{A}$ and $\vec{B}$ (also known as the dot product) is

$$\vec{A} \cdot \vec{B} = AB \cos \phi,$$

(6.1.1)

where $A$ and $B$ are the magnitudes of these vectors and $\phi$ is the angle between them. The dot product can be positive or negative depending on the sign of the cosine-function. It has the maximum value, when vectors $\vec{A}$ and $\vec{B}$ are parallel, which means the angle between them is zero and it becomes zero in the case of the perpendicular vectors.

By definition, the work done by the constant force $\vec{F}$ on the object which moves from point 1 to point 2 with displacement $\vec{d}$, is

$$W = \vec{F} \cdot \vec{d}.$$

(6.1.2)

Even though in every day life we apply word "work" to every physical or mental labor, in physics we have a nonzero work performed only if there is a displacement associated with the applied force. For instance, if you just hold something in your hands, even if it is very heavy, but you do not actually move the object somewhere, then you perform no work, because the object's displacement is zero. On the other hand, if you pull a heavy box along the floor, in that case you do perform work. If you choose the $x$-axis in the direction of the displacement then equation 6.1.2 becomes

$$W = F_x d,$$

(6.1.3)

which means that the work done on the object by a force during its motion is equal to the force’s component in the direction of the object’s displacement multiplied by the magnitude of this displacement. In order to move this heavy box along the floor, you
definitely have to apply the force in the direction you want it to move and the force applied in the perpendicular direction will produce no useful work.

The special unit is introduced in SI to measure work. It is called joule \((J)\) in honor of James Prescott Joule, an English scientist of 1800s. It is defined based on the equation 6.1.2 as

\[1 \text{ joule} = 1 J = 1 N \cdot 1m.\]

There are two important things to notice about definition of work 6.1.2. The first is that this definition is only applicable to the case of the constant force. We shall consider the case of a variable force later. The second thing is that this definition works for point-like objects only (or rigid bodies with all their parts moving in the same direction, no rotation).

One can see from the work's definition 6.1.2 or 6.1.3, that it can be positive as well as negative. This depends on the angle between the directions of force and displacement in equation 6.1.2 or on the sign of the force's component in the equation 6.1.3. The force does positive work, if it has the positive component in the direction of the displacement and it does negative work, if it has the component in the direction opposite to the displacement. The physical significance of this sign is that if the work is actually performed by this force, it is positive, while if it is performed against the force it is negative. Indeed if you push a heavy box along the floor you are performing positive work. You may ask where this work goes? It goes to overcome friction, so at the same time the friction force, which is always in the direction opposite to the direction of motion, performs negative work. If there are several forces acting on the object, we can talk about net work done by these forces. To calculate this work one has to find the work performed by each of the forces and then add all the works algebraically.

**Example 6.1.1.** What is the net work done on the box pushed with constant speed along a flat surface?

In this example there are 4 forces acting on the box. These are the pushing force of the person who moves the box, the frictional force between the box and the surface which has the direction opposite to the direction of motion, the gravitational force, and the normal force of the floor. Since the box is moving on the flat surface, the gravitational force and the normal force are perpendicular to the box's displacement, so, according to
equation 6.1, they perform no work. The pushing force, $\vec{F}_{\text{push}}$, is in the direction of motion and performs positive work $W_{\text{push}} = F_{\text{push}}d$, since it has the same direction as the displacement vector $\vec{d}$. The frictional force, $\vec{F}_f$, is directed opposite to the $\vec{d}$-vector and it performs negative work $W_f = -F_fd$. Let us notice that this box is moving with constant speed which means that it has zero acceleration. Then, according to the Newton's second law, we have $\sum F_x = F_{\text{push}} - F_f = ma_x = 0$, so $F_{\text{push}} = F_f$ and the net work done on the box will be $W_{\text{net}} = W_{\text{push}} + W_f = F_{\text{push}}d - F_fd = (F_{\text{push}} - F_f)d = 0$. So, if the box is moving with constant speed, the net work done on this box is zero.

2. Kinetic energy

As we saw in the previous example the net work done on the box moving with constant speed is zero. This statement is true not just for this box but for any object moving with constant speed. As it is stated earlier, work is characteristic of motion, so it is associated with the energy transferred to or from the object by means of the force acting on this object. If the object moves with constant speed there is no net force acting on it, so there is no work performed and there is no energy transfer. This means that the type of the energy associated with a motion should depend on the object's speed and it only changes if the object changes its speed.

Now let us consider the same box pushed by the person along the floor, but let the person apply the pushing force which is larger than the force of kinetic friction. In this case the net force acting on the box will not be zero and the box will accelerate according to the Newton's second law. The original speed of the box at point 1 was $v_1$, after it passes distance $d$ (in positive x direction) with acceleration $\vec{a}$ it will have the final speed $v_2$. We can find this speed from kinematics, which gives

$$v_2^2 = v_1^2 + 2ad.$$  

If we multiply this equation by the mass of the box and divide it by two, it gives

$$\frac{mv_2^2}{2} = \frac{mv_1^2}{2} + mad.$$
Let us notice that according to the Newton's second law the net force acting on the box is 
\[ F_{\text{net}} = ma \], and the work done by this force is 
\[ W_{\text{net}} = F_{\text{net}}d = mad \], so the last equation becomes 
\[ K_2 - K_1 = W_{\text{net}} \],
where we have introduced a new quantity \( K \), which is 
\[ K \equiv \frac{mv^2}{2} \]. (6.1.4)

The work done by the net force should change the energy of the object associated with its motion. Now we see that this work equal to the change of the quantity \( K \), which is known as the kinetic energy of the moving object. It can be seen from equation 6.1.4 that kinetic energy has the same units (joules) as work.

So, we have 
\[ \Delta K = K_2 - K_1 = W \], (6.1.5)
where for simplicity we have dropped subscript \( \text{net} \) near work. This last equation says that the change of the object's kinetic energy is equal to the net work done on the object or that the final kinetic energy of the object is equal to its original kinetic energy plus the work done on the object. Both those equivalent statements are known as the work-kinetic energy theorem. This statement stays true for both cases of positive as well as of negative work. In the first case the object's kinetic energy will increase, in the second case it will decrease under the influence of the net force. For example, if the pushing force acting on the box is larger than the frictional force, the net work is positive, the box is accelerating and its kinetic energy is growing. If the person will stop pushing the box, the only force left will be the frictional force, producing negative work, so the box will slow down and its kinetic energy will decrease until it stops. This means that all of the box's kinetic energy is gone into work performed against the frictional force. This shows that the work done by the frictional force on the object is always negative.

3. Work and kinetic energy in the case of three dimensions and variable force

In the example about the box which we were considering in order to derive work-kinetic energy theorem, the pushing force and the frictional force were acting along the same direction which was chosen as the \( x \)-axis and the box was moving in this direction as well. So, this was one-dimension problem. Moreover, we originally defined work in
such a way that the force was constant. This was true for both pushing force and frictional force in that example. Now we have to generalize our conclusions to the case of several dimensions and variable force.

Let us start from the case when the force is changing in magnitude, but still stays in the same direction in one dimension. So, the force and the displacement are directed along the same axis $x$. For instance, we can push this box but keep pushing it stronger and stronger increasing the force. In this case the applied force is a function of the box's position $x$, so $F = F(x)$. Assuming that this force is changing smoothly without sudden "jumps" and "downs", we can say that for the small enough change $\Delta x$ of $x$, the force almost stays constant, so the elementary work performed is $\Delta W = F \Delta x$. In fact, this is only true in the case when $\Delta x$ is infinitely small so we can introduce elementary work which is

$$dW = F \Delta x.$$  

(6.1.6)

To obtain the total work as the box's position is changing from $x_1$ to $x_2$, we have to find the sum of all elementary works $\Delta W$ or, in the limit of infinity small $\Delta x$, take the integral from the equation $6.1.6$ which is

$$W = \lim_{\Delta x \to 0} \sum \Delta W = \lim_{\Delta x \to 0} \sum F \Delta x = \int_{x_1}^{x_2} F(x) dx.$$  

(6.1.7)

Equation $6.1.7$ also shows that this work has a significance of the area on the $F$-$x$ plane under the curve $F(x)$ in between lines $x = x_1$ and $x = x_2$.

Suppose now that we apply force $\vec{F}(\vec{r})$ not in the direction of the box's motion, but in some arbitrary direction. Actually in most part of cases the box is pushed not exactly along the floor but at some angle with respect to the floor. There is no need to always choose displacement along the $x$-axis, it can be in any direction. The work was originally defined as the scalar product of the force vector and the displacement vector. This means that the elementary work in the case of the elementary displacement

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k},$$  

(6.1.8)

and arbitrary force,

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k},$$  

(6.1.9)

will become
\[ dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz. \] (6.1.10)

So the total work done by the force \( \vec{F} \) is

\[ W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz, \] (6.1.11)

which is a path integral from the force \( \vec{F} \) along the object's path described by its radius vector \( \vec{r} \).

Let us calculate the work done during the time interval \( \Delta t = t_2 - t_1 \). We can replace the variable in equation 6.1.11 by multiplying and dividing over \( dt \), so

\[ W = \int_{t_i}^{t_f} \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_{t_i}^{t_f} m\vec{a} \cdot \vec{v} dt = m \int_{t_i}^{t_f} \frac{d\vec{v}}{dt} \cdot \vec{v} dt = \]

\[ m \int_{v_i}^{v_f} \frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = K_2 - K_1 = \Delta K, \]

where we have used Newton's second law \( \vec{F} = m\vec{a} \), the definition of acceleration \( \vec{a} = \frac{d\vec{v}}{dt} \) and assumed that the mass of the box is constant. This derivation results in the work-kinetic energy theorem for the general case of any force and any displacement.

4. Work done by some important forces

Let us start from the most common gravitational force. We will consider the same box but in the case when it is thrown it into the air (maybe it was not so heavy after all). We already know from kinematics how the behavior of the box will look like in such a case. It will move up slowing down under the influence of gravitational force then it will stop at the highest point of its trajectory and fall down back to the ground. The original kinetic energy of the box was \( K_1 = \frac{mv_1^2}{2} \), where \( v_1 \) is its original speed. On the box's way up, when it rises to height \( h \), the work done by the gravitational force, \( mg \), is

\[ W_g = mgh \cos 180^\circ = -mgh. \]

So, the gravitational force does negative work on the box's way up and the box’s kinetic energy is decreasing. The box finally stops at the highest point with zero kinetic energy. After that it falls down, now the gravitational force and displacement are in the same direction, so

\[ W_g = mgh \cos 0^\circ = mgh. \]
This work is positive and has the same magnitude as before, so the kinetic energy of the box will increase and finally it comes back to the ground with the same kinetic energy and the same speed as it started. This all is only true if we ignore air resistance, otherwise we shall also include the work done by this air resistance force.

We can also consider a similar example, when the box is not thrown in the air but lifted at the same height $h$. There are two forces acting on the box in this case: the applied force $\bar{F}$ of the person who is lifting the box and the same gravitational force $mg$. The lifting force is directed upwards, while gravitational force is directed downwards. The kinetic energy change for the box will be

$$\Delta K = W_F + W_g,$$

the sum of the two works done by both forces. In the most part of cases, when the box is lifted at height $h$ it comes to rest at the end and it also was at rest on the ground before the lifting. This means that kinetic energy of the box does not change during the process and

$$W_F + W_g = 0,$$

$$W_F = -W_g,$$

the work done by the applied force of the person is equal to the work of gravitational force taken with the opposite sign. On the box's way up, when gravitational force performs negative work, trying to decrease the box's kinetic energy, the person performs positive work increasing the box's kinetic energy, so the final energy stays the same. On the box's way down, gravitational force performs positive work trying to increase box's kinetic energy, while the person who holds the box does not allow it to fall down decreasing its kinetic energy and performing negative work.

Gravitational force is constant everywhere near the earth's surface, so this was an example of the work done by a constant force. Now let us consider an example of the work performed by a variable force. The perfect example of such a force would be a force of the spring acting on the same box. Let us consider this box on horizontal floor attached to the originally relaxed (neither compressed nor stretched) horizontal spring. Another end of this spring is attached to the wall. If the person starts pulling or pushing the box, he/she will compress or extend the spring. If we choose direction of the $x$-axis along the displacement vector with the origin at the spring’s equilibrium position, the Hooke’s law gives us
This force gives us an example of a variable force which changes with changes of \( x \). Let us find the work done by this spring when it is either compressed or stretched by the person who pulls the box. The work of the spring between the two spring’s positions at points 1 and 2 is

\[
W_{s12} = \int_{x_1}^{x_2} F(x)dx = \int_{x_1}^{x_2} (-kx)dx = -\frac{kx^2}{2}\bigg|_{x_1}^{x_2} = \frac{kx_1^2}{2} - \frac{kx_2^2}{2}.
\]

This work is positive if \( x_1 > x_2 \), the original position of the spring was at larger distance from the equilibrium than the final position. In this case the spring’s force performs positive work, because the displacement and the force are both directed back to the equilibrium and kinetic energy of the spring is increasing. The work is negative if \( x_2 > x_1 \) and the spring moves away from equilibrium, so the force tries to slow down this motion.

If the spring was removed from equilibrium position then

\[
W_s = -\frac{kx^2}{2},
\]

where \( x \) is the final position of the spring and this work is negative.

For the work of the applied force of the person, we still have the same relation as before

\[
\Delta K = W_s + W_F.
\]

If the final and the original state of the box was at rest, then

\[
W_F = -W_s,
\]

so the work of the applied force is spent against the force of the spring. When the person removes the box from equilibrium he/she performs positive work, when he/she brings it back to equilibrium his/her work is negative.

5. Power

Sometimes work has to be performed during limited amount of time. For instance, some building materials of certain mass have to be lifted to the top of the building. In this case not only work matters, but the rate at which this work can be performed. This time rate at which the work is done by the force is said to be power. One can define the average power due to the force acting during the time interval \( \Delta t \) as

\[
F = -kx.
\]
\[ P_{\text{avg}} = \frac{W}{\Delta t}. \quad (6.1.14) \]

One can also define \textit{instantaneous power} as
\[ P = \frac{dW}{dt}. \quad (6.1.15) \]

From these definitions we can see that the SI unit of power is \textit{joule per second}. This unit has a special name watt (W), after James Watt, who greatly improved the rate at which steam engines could perform work. Another often used unit for power is the \textit{horsepower}.

The relationship between these units is
\[ 1 \text{ watt} = 1 \text{W} = 1 \text{J/s} = 0.738 \text{ ft} \cdot \text{lb/s}, \]
\[ 1 \text{horsepower} = 1 \text{hp} = 550 \text{ ft} \cdot \text{lb/s} = 746 \text{W}. \]

If we consider the work done by a constant force on a particle-like object, for instance if we are pushing the box with constant force along the straight line, we can relate velocity and power as
\[ P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{x}}{dt} = \vec{F} \cdot \frac{d\vec{x}}{dt} = \vec{F} \cdot \vec{v}. \quad (6.1.16) \]

\textbf{6. Potential Energy}

As we have learned, kinetic energy is the form of energy which is associated with the moving object, so if the object is at rest its kinetic energy is zero. We have also talked about the energy transfer occurring during the motion, when the work is done on the object by the force and this work goes either into increase or into decrease of the object’s kinetic energy. You could notice from the examples which we have considered, that even if the object is at rest having zero kinetic energy, it still can be potentially capable of performing work. For instance, if we consider a block of mass \( m \) at the top of the building of height, \( h \), this block has an ability to start motion falling down from this building. The same is true about the block, which is attached to the compressed spring. If we release this spring then the block will move. Here we are faced with a different form of energy, which is not the kinetic energy, associated with the block's motion at non-zero speed, but the \textit{potential energy}, \( U \), which depends on configuration (or arrangement) of the system. In the previous examples this dependence was either on the building's height or on the spring's compression. This form of energy is associated with potential ability of the system to perform work.
The first type of potential energy, mentioned in the example, is gravitational potential energy. This type of potential energy is related to separation distance between the objects. By changing this distance (between the block and the ground), lifting the block to the top of the building, one changes the system’s configuration and as a result it’s gravitational potential energy.

Another type of potential energy is elastic potential energy, which is associated with the state of compression or extension of the elastic spring. If we compress or stretch the spring that will change location of the different parts of this spring, so the work is performed to change its potential energy.

Let us now return to the example about the box thrown into the air. We have already proved that gravitational force performs the negative work on this box, while it is moving upward until it stops at the highest point of its trajectory. This work is negative because it reduces kinetic energy of the box. Where this lost energy will go? It goes into potential energy of the box-earth system. Then the box falls down and its kinetic energy is growing. Where this energy comes from? It comes from the potential energy of the box-earth system. It is transferred from there by means of the positive work done by the gravitational force.

So, we can define the change of the gravitational potential energy, $\Delta U$, of this system as the negative of work, $W$, done by the gravitational force

$$\Delta U = -W.$$  \hspace{1cm} (6.1.17)

This equation 6.1.17 is very general. It is valid not only for gravitational potential energy of the box-earth system, but it is also valid in the case of the box-spring system considered earlier.

*Exercise: Follow the same logical steps to prove equation 6.1.17 for the spring-box system.*

Let us highlight the key elements of the situations which we have just discussed

1) The system consists of two or more objects.

Indeed, in both cases we had at least two objects (the box and the earth or the box and the spring), however, it can be even more than two objects in the system. We can consider systems in several dimensions, attaching many springs to be box from different
sides or we can consider gravitational interaction for a system of several stars and planets.

2) The force acts between the particle-like object (the box in our examples) and the rest of the system.

So, we can say that this object is under the influence of the potential field from the rest of this system.

3) When configuration of the system is changing, the force does work on the particle-like object, transferring energy between the kinetic energy of the object and the potential energy of the system.

4) When configuration of the system is reversed, the force reverses the energy transfer by performing the same in absolute value but opposite in sign work bringing the system back to its original energy state.

If we denote the work done by the force, when we first changed configuration of the system as, \( W_1 \), and when the system is brought back to its original state as, \( W_2 \), then the last statement means that \( W_1 = -W_2 \). Forces, for which this relation is true, are called \textit{conservative forces}. Gravitational and elastic forces are both examples of conservative force. Forces for which this work relation is not satisfied are called \textit{nonconservative forces}. There are many examples of nonconservative forces. The most common example of such a force is the force of friction. Indeed every time, when the force of friction performs work, this work is always negative, because this force is always directed opposite to the velocity-vector. So, if we move a box along the floor and then move it back to its original position, even though configuration of the system has not changed but the total work done by the frictional force is not zero, it is negative. This means that mechanical energy of the box was reduced. It was transferred to the other form of energy, known as thermal energy. You may notice that the temperature of the floor and the box's surface has increased. There is no way to bring this thermal energy back to mechanical energy by decreasing temperature. This is why such an energy transfer made by means of work of nonconservative force is irreversible.

The existence of conservative forces provides a significant simplification to solution of some problems. Let us consider a particle moving along the closed path beginning at some initial position and then coming back to the same position. The shape
of this path (particle’s trajectory) does not matter. This fact provides significant simplification to our consideration. We shall call the force acting on the particle conservative if the total energy transferred to and from the particle during this round trip along this or any other closed path is zero. Or in other words: *The net work done by a conservative force on a particle moving around every closed path is zero.* We know from experiment that this condition works for both gravitational and elastic forces, so they are examples of conservative force. The important consequence of this condition is that *the work done by conservative force on a particle moving between the two points does not depend on the path taken by the particle.* So, if a particle moves in the field of conservative force from point \( a \) to point \( b \), whatever path it chooses, the work done by conservative force on this particle on both paths is the same as long as points \( a \) and \( b \) are the same:

\[
W_{ab,1} = W_{ab,2}.
\]  

(6.1.18)

This allows us to solve many problems involving very complicated paths by calculating work along much simpler path between the original and the final points. To prove this statement is a very easy task. Indeed, if the work done by the force along the closed path in conservative field is zero, then one can consider a particle moving from point \( a \) to point \( b \) along the path 1 and then back to \( a \) along the path 2. In this case we have

\[
W_{ab,1} + W_{ba,2} = 0,
\]

and the last equation becomes the same as equation 6.1.18.

For those who are familiar with calculus, the fact that work of conservative force along the closed path is equal to zero means that the scalar function (potential) \( U(\vec{r}) = U(x, y, z) \) exists, such that

\[
\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r}).
\]  

(6.1.19)

Symbol \( \vec{\nabla}U \) means “gradient of \( U \)”, which is a vector having components

\[
\nabla U_x = \frac{\partial U}{\partial x}, \quad \nabla U_y = \frac{\partial U}{\partial y}, \quad \nabla U_z = \frac{\partial U}{\partial z}.
\]

Let us calculate the work done by this conservative force, when the object’s position is changing from \( \vec{r}_1 \) to \( \vec{r}_2 \) which is
7. **Special types of potential energy**

Let us start from the most common example of potential energy, the gravitational potential energy near the earth's surface. We shall consider a particle of mass $m$ moving along the vertical axis $y$. This is the example of one-dimensional motion. The positive direction of axis $y$ is up, gravitational force is acting down, so it has the negative component on axis $y$ and we have

$$
\Delta U = -\int_{y_1}^{y_2} \left(-mg\right) dy = mg\int_{y_1}^{y_2} dy = mg(y_2 - y_1) = mg\Delta y .
$$

(6.1.24)

In fact, only the change of potential energy has physical significance, but it is convenient to use not just the change of potential energy, but also introduce the potential energy itself for every configuration of the particle-earth system. For this purpose, one can choose the reference configuration at which the potential energy is set to zero. Usually but not necessarily, the reference point is taken at the ground level ($y=0$), so then gravitational potential energy will be

$$
U(y) = mgy .
$$

(6.1.25)

This tells us that gravitational potential energy depends on the vertical position of the particle but not on its horizontal position.
The second example of conservative force is the force of the compressed or stretched spring. The elastic potential energy associated with this force can be found if we consider a block-spring system placed along axis $x$. The force acting on the block from the spring is directed back to equilibrium (negative $x$ direction), so the change of the potential energy is going to be

$$
\Delta U = -\int_{x_1}^{x_2} (-kx)dx = k\int_{x_1}^{x_2} xdx = \frac{kx_2^2}{2} - \frac{kx_1^2}{2}.
$$

(6.1.26)

The convenient choice of the reference configuration is the undeformed spring ($x=0$), so the elastic potential energy of the block-spring system is

$$
U = \frac{kx^2}{2}.
$$

(6.1.27)

8. Conservation of energy

The mechanical energy of the system is defined as the sum of its potential energy and its kinetic energy, which is

$$
E_{mec} = K + U
$$

(6.1.28)

Let us consider the closed system, such as that no external forces are acting on it and the only forces acting inside of this system are conservative forces. So, this system does not involve any type of friction or air resistance. When these conservative forces perform work inside of the system, they transfer energy from one state to another state. According to the work-kinetic energy theorem, the change of the system's kinetic energy is equal to the net work done, which is

$$
\Delta K = W,
$$
on the other hand in the case of conservative force

$$
W = -\Delta U,
$$
so we have

$$
\Delta K = -\Delta U,
K_2 - K_1 = -(U_2 - U_1),
K_2 + U_2 = K_1 + U_1,
E_{mec,2} = E_{mec,1}.
$$

(6.1.29)

This last equation is known as the principle of conservation of mechanical energy, which says that in the isolated system where only conservative forces cause energy changes, the
kinetic energy and potential energy can change, but their sum, the mechanical energy of the system, cannot change. We can also write this in the most compact form as

$$\Delta E_{mec} = \Delta K + \Delta U = 0.$$  \hspace{1cm} (6.1.30)

Example 6.1.2. Let us now return to the problem mentioned at very beginning of our discussion of work and energy. There is a car, which starts from the top of the hill with height $h$ above the ground. The car starts with speed $v_1$ and goes down the hill. What is the speed of the car at the bottom of the hill if there is no friction between the car's tires and the road? (Let us say it was during wintertime and the entire hill was covered with ice). To solve this problem using Newton's second law, we need to know the shape of the hill and if this shape is not a simple incline the solution becomes quite complicated. On the other hand we can use the law of energy conservation which we just derived. Let $m$ be the mass of the car, then the mechanical energy of this car at the top of the hill is

$$E_1 = K_1 + U_1 = \frac{mv_1^2}{2} + mgh.$$  

At the bottom of the hill the car has the speed $v_2$, which is what we are looking for, but the height is zero, so its total mechanical energy is going to be

$$E_2 = K_2 + U_2 = \frac{mv_2^2}{2} + mg(0) = \frac{mv_2^2}{2}.$$  

Since the only force acting on the car is the conservative force of gravity, the energy conservation works, which is

$$E_1 = E_2,$$

$$\frac{mv_1^2}{2} + mgh = \frac{mv_2^2}{2},$$

$$\frac{v_1^2}{2} + gh = \frac{v_2^2}{2},$$

$$v_2^2 = v_1^2 + 2gh,$$

$$v_2 = \sqrt{v_1^2 + 2gh}.$$  

So, we have found the car's speed at the bottom of the hill. It does not depend on the car's mass, neither it depends on the hill's shape.
Let us now consider a particle, which moves in one dimension. Let us say along axis $x$. For instance, it can be a small block at the end of the spring. Let us say we know how its potential energy depends on its coordinate, which is the known function $U(x)$. The knowledge of this dependence can provide a lot of useful information. In the case of only one conservative force acting on the particle in the same direction $x$, equation 6.1.19 becomes

$$ F(x) = -\frac{dU(x)}{dx}. $$

(6.1.31)

So, we can find the force as function of the $x$-variable, which is the negative slope of the $U(x)$ curve. If we also know the mechanical energy at some point, we can find the kinetic energy in all other points, since energy is conserved. In general we have

$$ K(x) = E_{\text{mec}} - U(x). $$

(6.1.32)

This kinetic energy can only be positive. This means that there are certain points on the energy curve, such that the particle cannot go beyond these points. It simply does not have enough energy. Those points are known as the turning points, because when the particle reaches these points its total energy becomes just potential energy, the particle stops and then goes back. For example, we can consider oscillations of the block attached to the spring. It never passes the distance larger than the original displacement from equilibrium. At the turning point the force acting on the particle has its maximum value, this is why the particle does not stay there even having zero velocity but comes back to its original position. On the other hand, the potential energy curve may have the maxima and minima values at some points. At these points the force acting on the particle is zero, since it is derivative of the potential energy. This is why we talk about those points as the equilibrium points. If the equilibrium position is associated with the minimum of potential energy then the particle will have maximum kinetic energy and maximum speed passing this position. However, the force acting on the particle near this potential energy minimum is always directed back to the equilibrium, so the particle returns. This is why we call this to be a stable equilibrium. In opposite if we have an equilibrium position associated with the maximum of the potential energy, in this case the force is always directed away from such equilibrium, so it is an unstable equilibrium. Finally, if one has
a horizontal part on the $U(x)$ curve, the derivative is zero and there is no force acting on the particle. This is the example of the *neutral equilibrium*.

Until this point we only considered closed systems, such as no external force was acting on it. Now let us consider an example, where external forces are involved. For instance, we can consider the closed system of the box and the earth and the external force of the person, who lifts this box to the top of the building. If there is no person, then the total mechanical energy stays the same. If the box is placed on the ground, it has zero energy and cannot jump to the top of the building. If the box is at the top, it can be dropped down, so its potential energy will be transferred to its kinetic energy. Now if the person lifts the box from the ground, he/she performs work. This work goes to the box's kinetic energy, since it is moving and to the box's potential energy since it is rising to the top of the building, so

$$W = \Delta K + \Delta U = \Delta E_{\text{mec}}.$$  

(6.1.33)

The work of the external force has been transferred to mechanical energy of the system.

Now, let us consider the case, when friction is also involved in the problem. Let the person push the box along the floor, where the force of kinetic friction is acting. In this case that force performs a negative work equal to the force of friction multiplied by the box's displacement. We can add this work to the left-hand side of equation 6.1.33, but since it is negative, we can move it to the right-hand side of the equation with the positive sign. This work increases the system's thermal energy, since both the box and the floor will be heated, so $\Delta E_{\text{th}} = -W_f$ and

$$W = \Delta K + \Delta U + \Delta E_{\text{th}} = \Delta E_{\text{mec}} + \Delta E_{\text{th}}.$$  

(6.1.34)

The last equation represents the law of conservation of energy, which is that the *total energy of the system can change only by the amounts of energy that are transferred to or from the system*, so

$$W = \Delta E.$$  

(6.1.35)

Here $E$ is the total energy of the system including all forms of energy. If there is no work done by external forces, then the total energy $E$ is conserved.

The definition of instantaneous power given before now becomes

$$P = \frac{dE}{dt},$$  

(6.1.36)
Example 6.1.3 Let us continue with the same car going down the icy hill, but now after it passes the hill it travels for some distance on the leveled ground with coefficient of kinetic friction $\mu$ after which it stops. What is this distance?

The car stops due to the fact that work is performed by the frictional force, which means that all of the car's mechanical energy is transferred to the thermal energy by means of this work. For the change of the thermal energy on the leveled ground we have

$$\Delta E_{th} = -W_f = -(F_f d) = \mu N d = \mu mg d,$$

where $d$ is the unknown distance. The original energy of the car was, as we remember

$$K_1 + U_1 = \Delta E_{th},$$

$$\frac{mv_1^2}{2} + mgh = \mu mg d,$$

$$d = \frac{v_1^2}{2\mu g} + \frac{h}{\mu}.$$