Lecture 4.5
Electromagnetic Induction

During last several lectures we have been talking about Magnetism. We have introduced the concept of magnetic field and saw that magnetic field is closely related to electric current.

Every electric current produces magnetic field as well as electric currents (or moving charges) are affected by external magnetic fields. The magnetic force $\vec{F}$ acting on a moving charge $q$ depends on the direction of the external magnetic field $\vec{B}$ relative to the direction of the charge’s velocity $\vec{v}$. Mathematically one can write that as

$$\vec{F} = q\vec{v} \times \vec{B}, \quad (4.5.1)$$

cross product of the two vectors.

Generalizing this equation for a current-carrying wire of length $L$ ($\vec{L}$ is a vector in the direction of current), we have calculated the force acting on the wire as

$$\vec{F} = I\vec{L} \times \vec{B} \quad (4.5.2)$$

Electric current itself produces magnetic field. In the case of the straight infinitely long wire we have proved that this magnetic field is proportional to the current $I$ in the wire and inversely proportional to the distance $r$ from the wire, so

$$B = \frac{\mu_0 I}{2\pi r}, \quad (4.5.3)$$

where constant $\mu_0 = 4\pi \times 10^{-7} \frac{Tm}{A}$ is known as permeability of free space.

We have also found that magnetic field produced at the center of the closed circular loop of radius $R$ carrying electric current $I$ is

$$B = \frac{N\mu_0 I}{2R}, \quad (4.5.4)$$

where $N$ is the number of wire turns in the loop.

If electric current produces magnetic field, is this possible to have a reverse process where an electric current is produced by the external magnetic field? Yes, it is possible. But it turns out that it is not that simple as one may think. Experiments show that constant magnetic field produced by a steady current can not create electric current in the other stationary coil.
The usual set up for this experiment is to have first coil (which is called primary circuit) with a current. This coil has several turns around the iron bar. The second coil (part of the secondary circuit) is turned around the same bar, but is not connected to the first coil. Experiments show that the only time when current appears in the secondary coil is when the switch in the first coil is in the process of closing or opening. This means that a current in the secondary circuit only appears if the current in the primary circuit is changing from zero to a steady value or vice versa. This was originally discovered by Michael Faraday, the 19th century British physicist-experimentalist.

You may conduct another experiment of the same nature. Instead of changing current in the primary coil you just move a permanent magnet through the secondary coil. In both those situations we have the number of magnetic field lines changing through the surface of the secondary coil. So, one can conclude that, in order to have an induced current or induced emf in the loop of wire, one has to change magnetic flux through the surface of this loop. Let us introduce the exact mathematical definition of magnetic flux, which is very similar to definition of electric flux. Magnetic flux is related to the number of magnetic field lines crossing the surface of the coil. It is easy to see that for a flat loop

\[ \Phi = BA \cos \theta = \vec{B} \cdot \vec{A}, \]  

where \( A \) is the area of the coil, \( B \) is the absolute value of magnetic field and \( \theta \) is the angle between the normal to the surface of the coil and magnetic field. The SI unit of magnetic flux is Weber (1Wb = 1T * 1m²). If the coil is not flat then, as in the case of the electric field

\[ \Phi = \int \vec{B} \cdot d\vec{A}, \]  

Summarizing the results of the different experiments, we can obtain the Faraday’s law, which states that an electromotive force is induced in a circuit when there is a changing magnetic flux passing through the surface of the loop of the circuit, and it is equal

\[ \varepsilon = -\frac{d\Phi}{dt}, \]  

to the change of magnetic flux per unit of time. The minus sign has to do with direction of current in the circuit and we will discuss it in a few moments. This process is also called electromagnetic induction. If the coil consists not of one but of several turns, the equation 4.5.7 has to be multiplied by the number of turns. Note that since magnetic flux involves
three different terms (equation 4.5.5) then emf will be produced if any of these terms is changing. So, one can either change magnetic field or area of the coil or rotate the same coil inside of the uniform magnetic field. We shall consider some of these situations in details.

The direction of the induced current generated by a change of magnetic flux produces magnetic field that opposes the change in the original magnetic flux. The last statement is known as the Lenz’s law.

The Faraday’s law forms the basis to explain how electric generator is working. It is essentially a device which transforms mechanical energy into electromagnetic energy and allows obtaining electric current by performing mechanical work, changing magnetic flux through the surface of the coil. It is usually done by rotating the current loop inside of the external magnetic field. If the loop is rotated with constant angular speed $\omega$ then the magnetic flux is

$$\Phi = AB \cos(\omega t),$$

since $\theta$ is now representing the angular position of the coil. This behavior is similar to simple harmonic oscillation, which we studied during last semester. We saw there, that if position behaves as a cosine-like function of time then velocity behaves as a sine-like function of time. The same situation occurs here giving us

$$\epsilon = -\frac{d\Phi}{dt} = -BA \frac{d\cos(\omega t)}{dt} = BA\omega \sin(\omega t), \quad (4.5.8)$$

So, this generator will produce emf, which is also changing with time, this means we have an example of AC (alternating current) generator.

The similar principle explains how transformer works. It usually consists of two coils with different number of loops (the same type of device, we considered in Faraday’s experiment) Changing voltage in the primary coil produces emf in the secondary coil. The ratio between the amplitudes of voltages is proportional to the ratio of the number of loops.

$$\Delta V_2 = \Delta V_1 \frac{N_2}{N_1}, \quad (4.5.9)$$

Even though voltage is different in both coils but this is not in contradiction with the energy conservation law, since the current is smaller in the coil with larger voltage.
Because of this the power stays the same in both coils and the transformer equation is the following

\[
\frac{I_2}{I_1} = \frac{\Delta V_1}{\Delta V_2} = \frac{N_1}{N_2}
\]  

Let us now see what happens if not the angle or magnetic field is changing but the area of the loop is. We can observe such a situation if we consider a rectangular loop of wire where one side of the loop is the metal rod, which is free to move relative to the two other sides. If one moves the bar with constant speed \( v \) and at the same time the loop stays in the region of space with uniform magnetic field \( B \), directed perpendicular to the surface of the loop, then

\[
\Delta \Phi = \Delta (BA) = B\Delta A = Bl v \Delta t ,
\]

where \( l \) is the length of the bar. So

\[
|\varepsilon| = \frac{d\Phi}{dt} = Bvl .
\]  

If total resistance of this loop is \( R \), then it will be current in the loop which is

\[
I = \frac{|\varepsilon|}{R} = \frac{vBl}{R} ,
\]  

Direction of this current is determined by the Lentz’s law, so the current will produce magnetic field, which opposes the change of the external magnetic flux through the coil.

Let us look at the energy transformations in this circuit. In order to keep this rod moving, one has to perform work against the external magnetic field. If the rod is moving with constant speed \( v \) has to apply the force equal to the magnetic force (equation 4.5.2). The mechanical power delivered by the external force is

\[
P = Fv = llBv = \frac{lvB}{R} llBv = \frac{(lvB)^2}{R} ,
\]

One the other hand the electric power will be dissipated in the resistor, which is

\[
P = I^2 R = \frac{(vBl)^2}{R} .
\]

So, we can see that conservation of energy does work in this system and all mechanical energy of the moving rod is transferred into electric energy dissipated in the resistor.

As one can see from all of the examples the changing magnetic flux always produces \( emf \) in conductors. In order for \( emf \) to exist, it should be an electric field, which
causes it. So, we can conclude that changing magnetic field produces an electric field. This electric field is different from the one we discussed before, first of all because there are no charges to produce it, second because this time the electric field lines are very similar to magnetic field lines, due to the fact that they are closed loops. If we now consider the work done by this electric field on the unit of the positive testing charge, on the one hand this work is equal to \textit{emf}, on the other hand it can be calculated in terms of electric field as

\begin{equation}
\varepsilon = \oint E \cdot d\vec{s},
\end{equation}

which allows us to rewrite Faraday’s law in alternative form

\begin{equation}
\oint E \cdot d\vec{s} = -\frac{d\Phi_B}{dt},
\end{equation}

We have stared our discussion of electromagnetic induction today by considering Faraday’s experiment with two electric circuits containing solenoids built on the same metallic bar. We saw that changing current in the primary circuit produces changing magnetic field and causes induced current in the secondary circuit. This is called mutual induction. But what happens in the primary circuit? Since its own magnetic flux is changing through its own surface, it should be the induced current in the primary circuit as well. This phenomenon is called self-induction. The direction of the self-induced current in the primary coil is determined by the Lentz’s law. So if one tries to increase current in the primary circuit then the self-induced current will be in the direction opposite to the original current. If one tries to decrease current in the primary circuit, then the self-induced current will be in the same direction as the original current.

Faraday’s law can still be applied to the primary coil as well as to the secondary coil, so the self-induced \textit{emf} can be calculated. We also know that the magnetic field involved into definition of magnetic flux is produced by the current in the coil and so it is proportional to this current. Then magnetic flux is also proportional to current in the coil. The coefficient of proportionality \( L \) between the current in the coil and its self-flux is called \textit{inductance}, so

\begin{equation}
\Phi = LI.
\end{equation}

Faraday’s law can then be written as
\[
\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}.
\] (4.5.16)

Inductance of the coil depends on its geometrical characteristics, but not on the current in it. We can also write equation 4.5.16 as

\[
L = \left| \frac{d\Phi}{dl} \right| \quad (4.5.17)
\]

From the last several equations we can see what the units for inductance are. This unit has its special name. It is called Henry (H). (1H = 1V*s/A).

Let us apply the equation 4.5.15 to calculate the inductance of an ideal solenoid. In this case the field is parallel to the axis of the solenoid (perpendicular to the cross section) and it is the uniform field \( B = \mu_0 nI \), where \( n \) is the number of turns per unit of solenoid’s length. So

\[
L = \frac{NBA}{I} = \frac{N\mu_0 N^2 IA}{l} = \mu_0 \frac{N^2}{l} A = \mu_0 n^2 Al, \quad (4.5.18)
\]

where \( l \) is the length of solenoid.

When we started our discussion of solenoids, we saw that solenoid does similar things for magnetic field as capacitor does for electric field. We remember that capacitor could serve as a storage place for electric energy. Probably solenoid should play the same role for magnetic energy. However, there are differences as well. In the case of the capacitor it was the storage of electric potential energy. Magnetic field, on the other hand, is not a potential field. Magnetic field has to do with currents or motion of charged particles. The energy associated with motion is the kinetic energy, so a solenoid may somehow play the role of the place, where kinetic energy of the moving charges can be stored. We even noticed that self-inductance prevents current from changes, so it tries to keep the same kinetic energy in the solenoid.

In the case of a capacitor, in order to store the charge, we have considered RC circuit. The analogy tells us that probably now we have to study RL circuit which consist of a resistor, battery and a solenoid (inductor).

Let \( R \) be the total resistance of the resistor and the inductor. If one closes the switch the potential gain in the system is going to be \( \varepsilon - L\frac{dI}{dt} \), where the first term is \( emf \) due to the battery and the second term is the self-induced \( emf \) in the inductor. This
potential gain will be equal to the potential drop $IR$, where $I$ is the current in the circuit. So that

$$\varepsilon - L \frac{dI}{dt} = IR$$

$$\frac{L}{R} \frac{dI}{dt} + I = \frac{\varepsilon}{R}.$$ 

The solution of this equation has an exponential form like

$$I = \frac{\varepsilon}{R} \left( 1 - \exp \left( -\frac{R}{L} t \right) \right). \quad (4.5.19)$$

If one wants to measure voltage across the resistor it will be

$$V_r = iR = \varepsilon \left( 1 - \exp \left( -\frac{R}{L} t \right) \right) \quad (4.5.20)$$

The time constant for this process is $\tau = \frac{L}{R}$ and one can define a half-life time in a same fashion as for the RC circuit $t_{1/2} = \frac{L}{R} \ln 2$. You could measure this time experimentally and compare to this prediction.

To find the voltage across the inductor let us recall that the total potential change in the system is equal to $\varepsilon$, so $\varepsilon = V_r + V_L$ or

$$V_L = \varepsilon - V_r = \varepsilon - \varepsilon \left( 1 - \exp \left( -\frac{R}{L} t \right) \right) = \varepsilon \exp \left( -\frac{R}{L} t \right) \quad (4.5.21)$$

If one opens the switch the potential change in the system going to be just $-L \frac{dI}{dt}$, which is due to the $emf$ in the inductor. This potential change will be equal to the potential drop $IR$. So that

$$-L \frac{dI}{dt} = IR$$

$$\frac{L}{R} \frac{dI}{dt} + I = 0.$$ 

The solution of this equation has an exponential form like

$$I = \frac{\varepsilon}{R} \exp \left( -\frac{R}{L} t \right) \quad (4.5.22)$$
where we have used the fact that current in the circuit was $I = \frac{\varepsilon}{R}$ before one opens the switch. If one wants to measure voltage across the resistor, it will be
\[
V_R = IR = \varepsilon \exp \left( -\frac{R}{L} t \right)
\]  
(4.5.23)

To find voltage across the inductor let us recall that $V_R + V_L = 0$, so
\[
V_L = -\varepsilon \exp \left( -\frac{R}{L} t \right)
\]  
(4.5.24)

Let us now think about energy of the magnetic field inside of the inductor. If one plugs inductor to the battery, then in order to increase current (overcome self inductance) the certain amount of work has to be performed by the battery. This work will go to increase energy of the magnetic field inside of the inductor. We can perform this calculation in the same they as we did for a capacitor. Since current through the inductor is changing, we have to integrate the current between its final value $I$ and original value of zero, so
\[
P = \frac{dU}{dt} = I \varepsilon = IL \frac{dI}{dt}
\]
\[
dU = I L dI
\]
\[
U = \frac{I^2 L}{2}
\]  
(4.5.25)

The energy stored in the inductor is then going to be
\[
U = \frac{I^2 L}{2}
\]  
(4.5.26)

If the inductor is an ideal solenoid then
\[
U = \frac{1}{2} \left( \mu_0 n^2 A l \right) I^2 = \frac{1}{2 \mu_0} B^2 A l .
\]  
(4.5.27)

The volume density of this energy (per unit of solenoid’s volume) is
\[
u_B = \frac{1}{2 \mu_0} B^2
\]  
(4.5.28)

Example 4.5.1. The magnetic energy of 13 J is stored inside of ideal solenoid, which has the length of 31 cm, the diameter of 7.0 cm and 500 turns of wire. What is a) current flowing through this solenoid, b) magnetic field inside of the solenoid, c) magnetic energy density inside of the solenoid?
Solenoid magnetic energy is $E = 13J$, it has $N^2$ 500 turns of $d = 7.0 \text{ cm}$.

Energy of the magnetic field is $E = \frac{1}{2} LI^2$

The inductance of the solenoid is $L = \frac{\mu_0 N^2 A}{2}$,

So $E = \frac{1}{2} \mu_0 N^2 A I^2$

$I^2 = \frac{2EE}{\mu_0 N^2 A}$

4) $I^2 = \sqrt{\frac{2EE}{\mu_0 N^2 A}} = \sqrt{\frac{2(13J)}{\mu_0 N^2 A}} = \sqrt{\frac{2(13J)(0.51m)}{4\pi \times 10^{-7} Tm/A (500)^2 \times (0.07m)^2}} = 80A$

5) $B = \mu_0 \frac{N}{A} I = 4\pi \times 10^{-7} Tm/A \times 500 \times 80A = 0.2 T$

6) $E = \frac{1}{2} LI^2 \frac{1}{AC} = \frac{1}{2} \mu_0 \frac{N^2 A I^2}{AC} = \frac{1}{2} \frac{\mu_0 (N^2 A I^2)}{C} I^2$

$= \frac{1}{2} (4\pi \times 10^{-7} Tm/A) (\frac{500}{0.51m})^2 (80A)^2 = 11 \text{ kJ/m}^2$