Lecture 4.6  
RLC Circuits  

During our discussion of electric and magnetic fields we have seen that the energy of the electric field as well as electric charge can be stored in a device, known as a capacitor. This energy a quadratic function of electric charge \( q \) on the capacitor’s plates:

\[
U_e = \frac{q^2}{2C},
\]

(4.6.1)

At the same time magnetic energy as well as electric current can be stored in a device, known as an inductor. That energy is a quadratic function of current \( I \) in the inductor:

\[
U_B = \frac{LI^2}{2},
\]

(4.6.2)

You may also notice that current, which appears in equation 4.6.2 is the time derivative of the charge, which appears in equation 4.6.1. So, the situation described here is very similar to the situation which takes place for a mass oscillating on a spring. Indeed, the potential energy of the spring is quadratic function of mass’s displacement:

\[
U = \frac{kx^2}{2},
\]

(4.6.3)

and the kinetic energy of mass is quadratic function of its velocity:

\[
K = \frac{mv^2}{2},
\]

(4.6.4)

where velocity is the time derivative of displacement in equation 4.6.3.

Due to complete mathematical analogy between the first and the second pair of the equations, the physical process described by them should also be similar. In particular, we know that during the process of mechanical oscillations of the mass attached to the spring, there is transfer of energy between the potential energy of the spring and kinetic energy of the mass. Obviously, it should be a way how one can set up a similar process, where the energy is transferred between electric and magnetic fields. This can be done if we first charge a capacitor and then connect it in series with an inductor. In such a case, the capacitor will be discharging through the inductor, so the charge and the potential energy of the electric field in the capacitor will be decreasing, but at the same time the current and the energy of the magnetic field in the inductor will be increasing. This will
continue until the current will reach its maximum value and at that moment the capacitor will be completely discharged. But, since we already have current going on in the circuit, this current will cause recharge of the capacitor until all the energy is again transferred and stored there. These oscillations of electric current and electric charge will continue forever unless the resistance of wires is taken into account, since some of the energy will be lost there in the form of heat.

To obtain the equation of motion for the charge, we can apply the Kirchhoff’s loop rule for the loop with inductor and capacitor connected in it. The law gives us

\[ V_c + V_L = 0, \]
\[ \frac{-q}{C} - L \frac{dI}{dt} = 0, \]
\[ \frac{q}{C} + L \frac{d^2q}{dt^2} = 0, \]
\[ \frac{d^2q}{dt^2} + \frac{q}{CL} = 0, \]

The last equation is equivalent to

\[ \frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \]

for the motion of mass on the spring. We already know that the angular frequency of that motion is \( \omega = \sqrt{\frac{k}{m}} \), so we can conclude that in the case of the equation 4.6.5, the angular frequency of oscillations of charge between the capacitor and the inductor is going to be

\[ \omega = \frac{1}{\sqrt{LC}}. \]

We can also conclude that the charge on the capacitor will behave as

\[ q = q_0 \cos(\omega t + \phi), \]

then the current is going to be

\[ I = \frac{dq}{dt} = -\omega q_0 \sin(\omega t + \phi). \]

In reality, of course, any wire has some resistance. This is why the oscillations described above are not going to continue forever, but instead they are going to decay.
This will be similar to the damped oscillator, which we discussed last semester. You may recall the solution, which should now have a form

\[ q = q_0 \exp\left(\frac{-R}{2L}t\right) \cos(\omega't + \phi), \]  

(4.6.10)

where \( \omega' = \sqrt{\omega^2 - \left(R/2L\right)^2} \). This solution shows that there is exponential decay on the top of oscillatory motion with slightly changed frequency.