Lecture 1.1
Introduction, Syllabus, Review of PHYS2510, Mechanical Waves

1. Syllabus

See the class syllabus at the class Web Page and/or Moodle.

2. Survey

Please complete the first-day survey offered through Moodle.

3. Introduction

You have spent a lot of time last semester talking about why it is so important to study physics.

*Please think about why do you study physics and why is this important?*

In my opinion, these are the reasons for studying physics:

- Physics is a fundamental science which provides the basis for all of the natural sciences such as chemistry, biology, engineering, geology, astronomy and many others. There is no way to understand any of these fields without knowledge of physics.
- Many devices we use in the modern technological world, even on every day basis, can not be operated without understanding of the basic principles of physics.
- Studying physics, as well as any other natural science, develops a way of logical thinking which is very important for any field of study.
- Physics challenges us with interesting nontrivial problems. To learn how to solve non-standard problems is one of the essential qualities of the future leader.
- Finally, studying physics is just interesting and it is fun.

You have already learned a lot of things about physics. Based on your experience with PHYS2510, please think of your own definition of Physics? Would you agree or disagree with my definition? My definition is that Physics is a science which looks for the basic laws of nature and then tries to understand how these laws are working. Please continue to think on this question, you may submit your definitions and comments through Moodle.

Laws of physics refer to the natural phenomena at different scales starting from the micro level of elementary particles and up to the macro level of the universe itself. During the last semester you have discussed laws of Physics as they apply to classical mechanical systems, consisting of objects with typical for everyday life sizes and moving at low (compared to speed of light) speeds. This semester we shall spend a lot of time talking about micro scale and motion of atoms and molecules.
Throughout human history numerous physics laws were discovered. But it turns out that at each successive step in the development of physics as a science, the basic laws and theories become simpler and fewer in numbers. Some natural laws happened to be just consequences of other more fundamental principles. The final goal of physics always was and is to find the universal law which will unify all the known phenomena and explain nature at all possible scales. This goal is still to be achieved.

You started the study of Physics last semester by learning basics laws of classical mechanics: Newton’s laws and various conservation laws. Now we have to find out if these laws were fundamental enough to be used in other areas of physics. 

*Please try to recall the statements of Newton’s laws and the laws of conservation of energy, momentum and angular momentum.*

Traditionally physics is divided into several fundamental subareas, such as Mechanics, Thermodynamics, Statistical Physics, Electromagnetism, Optics, Atomic Physics, Nuclear Physics and Theory of Relativity. Even though these phenomena look like completely unrelated, it has been discovered that connections between them are much closer than people originally thought. 

*Think about these connections and give some examples.*

For instance Optics, which has to do with nature of light, is tied very closely to Electromagnetism, since light is an example of electromagnetic wave. On the other hand the study of wave motion is the subject of Classical Mechanics. However, light is an unusual wave. It always moves with the same speed in vacuum and this have become one of the fundamental principles for the Theory of Relativity. You can find a lot of other interconnections between the areas of physics mentioned above. This confirms my comment about universal nature of Physics.

Being such a universal science, physics provides the basis for our understanding of other natural sciences such as chemistry, biology, astronomy, geology and so on. Each of the mentioned sciences of course can not be reduced to a sub-area of physics, but their understanding is not possible without knowledge of physics. Physics provides the basis for other material sciences in a same way as classical mechanics provides the basis for physics itself. Historically classical mechanics was the first area of physics developed and understood by people. It also was the first area of physics you studied in Phys 2510.

However, I should warn you from the common misunderstanding of the 18th-19th century scientists, who thought that all the workings of nature could be reduced to some kind of mechanical machine, which works only according to Newtonian principles. Mechanics is the essential part and
the basis of physics, but physics can not be reduced to mechanics. Classical mechanics only works at certain conditions and at certain scales. Since these conditions and these scales are the ones we live in during our everyday life, we use to project them on the entire world. But the surrounding world is much more complicated than that.

Hopefully you are already convinced that it is important to study physics, but how do we approach this task. Since physics is a natural science, we have to use the same Scientific Method as for any other natural science.

Please recall the basic steps of scientific method.

If you are trying to explain some natural phenomenon, first of all you will perform some observation of this phenomenon and then generalize your observations in the form of the empirical law. This law does not do any predictions or explanations. It is just generalization of visible facts. Then you will probably need to find out what has been already done to explain this phenomenon by other scientists. To do that you will go to the library and work with the literature available there. Based on what you have learned and your own observations you may introduce your own hypothesis to explain this phenomenon. To make sure your hypothesis is reasonable you need to look at other consequences following from it and check if they are confirmed by experiments. Now your hypothesis becomes a theory which you can discuss with the colleagues in your field. They may give you some suggestions for improvement as well as they can check predictions of your theory by performing their own experiments. Only after the theory has been confirmed by independent observations, it may become widely accepted and considered to be a valid explanation for the phenomenon which you have studied. We were trying to follow (as time allowed) this pattern in our study of classical mechanics, we will try to use the same approach this semester again.

So, as you can see physics is quantitative as well as qualitative science. Both these aspects are of equal importance. This means that physics can not be reduced to plug in numbers and check the answer mathematical exercise. This is because different problems have different qualitative nature, so you really have to understand why this or that equation has to be used for some particular case. On the other hand there is no way the problem can be solved without mathematics. This is because physics has to do with quantities. These can be measured during experiments. The answers for many questions in social sciences accept and even appreciate diversity of opinions. Physics only accepts such a diversity until certain measurement has been performed and the answer has been found with certain accuracy.
Physics has to do with material phenomena. These can be *a) verified by independent observes* and *b) measured in terms of physical quantities*. As you have noticed, several times we mentioned word *measurement*. What does it mean to measure something? From our experience in the lab we already know that to measure means to compare with a *standard*. Since there are many different quantities in nature, it should be many standards for these quantities. However, not all of the quantities are independent. From mechanics we already know that, for instance, velocity is length divided by time, density is mass divided by volume; volume itself is length multiplied by length multiplied by length. After all, there are only three independent quantities in mechanics, such as *length, time and mass*. All other quantities in mechanics can be measured using *derived units* based on these three basic dimensions. We shall see during this semester that there are some other fundamental dimensions in other areas of physics outside of classical mechanics but again not too many.

In mechanics we have introduced standards to measure the three basic quantities. Each standard corresponds to exactly one *unit* of quantity. In *international system of units (SI)* based on the French *Metric system*, the basic units are *Meter* (m) for length, *Second* (s) for time, *Kilogram* (kg) for mass. The units for other quantities in mechanics sometimes have their special names, sometimes do not, but all of them can be derived as results of some combinations of the basic units. For example, the unit of force has its own name Newton, but 1 Newton = 1 kilogram* 1 meter/ 1 second squared, the unit for momentum does not even have the name it is just 1 kg*m/s.

The choice of the basic units is dictated by internal properties of our universe, such as existence of space, time and matter. These are the basic elements and their properties have to be described by means of the basic dimensions. Mass is one of the essential but not the only one of attributes of matter. This semester we shall learn about other attribute, the electric charge. To measure it we shall use dimension of charge. Continuity of space and time is measured by means of dimensions of length and time.

Since we have already mentioned existence of different systems of units, one often needs to convert units from one system to another. This can be done with the help of *conversion factors* (a ratio of units that is equal to unity). Useful conversion factors can be found in Appendix D on the book.

As we said, physics is quantitative science, where everything can be measured in exact terms. This is only partly true, since no one measurement can be precise. The accuracy of a measurement depends on many factors, such as who was performing the measurement and how it was performed,
what types of measuring devices were used and so on. Each device has its own precision due to its constructional specific. Measuring errors may occur due to different random circumstances which may take place during the experiment. Conducting the experiment many times and taking the average of all the measurements one can eliminate this error, known as random error. On the other hand, it is not possible to eliminate an error due to the measurement device as long as one uses the same device for all the measurements, this error is known as systematic error. So every measurement has a certain amount of error, this is why every physical quantity has to be presented with limited number of significant figures. We have already talked about error analysis last semester. Please recall what we learned about it, since we will be using many of these techniques in the lab again.

4. Waves

At the end of the previous semester you have considered a subject of simple harmonic motion. It is an example of periodic motion in time. In that case position of the object was changing with time according to sinusoidal law. The total mechanical energy of the system was conserved (in the absence of damping), while potential and kinetic energies of the system were changing as a square of the sine-like function. Even though the object was moving periodically, the process does not involve any motion of energy in space. All the energy was concentrated in the system and it was not traveling anywhere except transferring from one form of mechanical energy to another form of mechanical energy inside of the oscillator.

Today we are going to consider a different periodic phenomenon, known as waves. This physical phenomenon provides an example of how the energy (or information) can be moved through space without actual motion of the matter.

Let us consider the following example. If you want to communicate with your friend who lives very far away, you can either send him/her a letter by a regular mail or you can use an e-mail to do the same thing. Now days many people prefer the later method, because it is much faster and more efficient. The first method involves actual motion of matter (the letter) to your friend. We are already familiar with this is the type of motion. The second way, however, does not involve motion of matter. Instead you are sending an electromagnetic wave, which only delivers energy and information but not a physical object. We will talk about electromagnetic waves later. Now we are going to study another type of waves, known as mechanical waves. The typical example of such a wave is a sound wave. We all know that we can pass information by means of sound and, at the same time, we are not actually passing some material objects from person to person.
Particle and wave are the two essential concepts in physics. The former is the tiny concentration of matter capable of transmitting energy and located at certain point in space, the later is a broad distribution of energy filling large areas of space but not having the exact location. Almost all areas of physics have to do with these two concepts. Moreover, as you will see later, sometimes it is not even possible to distinguish between a particle and a wave, as in the case of elementary particles. Those can demonstrate both types of properties depending on the problem.

The main types of waves existing in nature are

1) Mechanical waves. This is the most familiar for everyday life type of wave. We are dealing with them every day in the form of sound waves, waves on the surface of water and, sometimes as seismic waves. All these waves are governed by Newton's laws and they can only exist in the material medium, such as gas, liquid or solid.

2) Electromagnetic waves. These waves are also very common, but maybe less familiar. They include light waves of all types, radio waves, microwaves, radar waves and so on. These waves do not require any medium to exist. The universal property of these waves is that they all have the same speed in vacuum of \( c = 299792458 \text{ m/s} \).

3) Matter waves. These waves are associated with electrons, protons and other fundamental particles. They are studied in modern quantum mechanics and field theory.

The only type of wave we shall study today is the first type of wave: the mechanical wave.

A simple example of mechanical wave is the wave sent along the stretched taut string. This becomes possible, because the string is under tension. So, if the string is pulled up or down at one end, it then begins to pull up or down the adjacent segment of the string, then the next segment is pulled and so on. As a result a pulse will be traveling along the string with some velocity \( v \). If the person who pulls the string continues doing so periodically, the periodic wave will travel along this string. If the external force causing this motion has a form of the simple harmonic function then this wave will have a shape of sinusoidal function. Here we make an assumption that there is no friction in the string, so this wave will not die out while traveling, and the string is long enough, so we do not have to take into account the other wave traveling in the opposite direction as a result of the rebound from the opposite end of string. If we consider a motion of some element of this string, we shall see that it moves up and down in the direction perpendicular to the direction of propagation of this wave. This is why we shall call such a wave a transverse wave.

Transverse wave is not the only type of mechanical waves. As an alternative we can consider a wave produced by a piston in a long air-filled pipe. In a same way as the person was pulling on the
string, he/she can move the piston rightward and leftward. This motion of the piston can be set up as simple harmonic motion as well. At first it will cause the increase of air pressure right next to the piston. Then this pressure increase will move to the next section of the pipe and so on. When the piston moves back, the pressure will decrease, this decrease will also travel along the pipe. As a result we have a sound wave in the pipe. However, this wave is different from the one in the string, because now different elements of air in the pipe are moving in the same direction as the direction of wave’s travel. We shall call such a wave a *longitudinal wave*.

Both transverse waves and longitudinal waves are said to be traveling waves, because they both travel from one point in space to another point in space. It is the wave that moves but not the material. In both cases elements of the string or elements of the air in the pipe do not travel anywhere. They just oscillate around their equilibrium positions.

These waves are traveling at certain speed. If we are talking about solid substances, both transverse and longitudinal waves can exist there. Moreover, those waves have different speeds. This fact is often used to determine the unknown distance to some object. The longitudinal wave with speed \( v_1 \) is usually traveling faster, than transverse waves with speed \( v_2 \). Let \( d \) be the unknown distance to the object, which emits both types of waves. It will take time

\[
t_1 = \frac{d}{v_1}
\]

for the first wave to travel from the object to the observer and time

\[
t_2 = \frac{d}{v_2}
\]

for the second wave. The observer can measure the time \( \Delta t = t_2 - t_1 \), which passes between the detection of the two waves, so

\[
\Delta t = \frac{d}{v_2} - \frac{d}{v_1},
\]

\[
d = \frac{\Delta t v_1 v_2}{v_1 - v_2}.
\]

So, one can find the unknown distance.

Today we will concentrate our attention on the transverse waves. Let us consider a transverse wave in the string. This is an example of the plane wave, which propagates in the direction perpendicular to the plane. We shall choose this direction as the \( x \)-direction. Since it is transverse wave; the elements of the string will move in the direction perpendicular to the \( x \)-direction. We will
choose that direction as the \( y \)-direction. This means that position of the elements of the string will change with time and it will be different for different values of \( x \). To describe this wave we will need to know this function \( y = y(x,t) \) for displacement of the elements in the string. We shall only consider sinusoidal waves, behaving according to

\[ y(x,t) = y_m \sin(kx - \omega t). \tag{1.1.1} \]

In this equation \( y_m \) is called the wave’s amplitude, \( k \) is called the wave number, \( \omega \) is the wave’s angular frequency. The wave described by the equation 1.1.1 is moving in the positive \( x \)-direction.

Let us clarify the physical significance of all the constants in equation 1.1.1. Quantity \( y_m \) stands for the wave’s amplitude, because it is the maximum value which \( y \) variable can achieve. The \textit{phase} of the wave is the \textit{sine}-function’s argument \( kx - \omega t \). If we fix \( x \) in this argument, we can see how position of the string is changing at certain location in space. According to the equation 1.1.1, it oscillates in a simple harmonic motion. If we fix the value of time \( t \), we can see a snapshot of this wave in space. It will look like a \textit{sine}-function changing along the \( x \)-direction. It is the periodic function, which reproduces its shape along the \( x \)-axis. The smallest distance between the two repetitions of the wave’s shape along the \( x \)-axis is called a \textit{wavelength} \( \lambda \). Let us find how this quantity is related to other parameters in equation 1.1.1. According to our definition of the wavelength at certain moment in time one has

\[ y(x,t) = y(x + \lambda, t) \]
\[ y_m \sin(kx - \omega t) = y_m \sin(k(x + \lambda) - \omega t), \]
\[ kx - \omega t + 2\pi = k(x + \lambda) - \omega t, \]
\[ 2\pi = k\lambda. \]

This means
\[ k = \frac{2\pi}{\lambda}. \tag{1.1.2} \]

The SI unit for the wave number is radian per meter (\textit{rad/m}).

Now let us consider how the position of the string’s element changes with time at certain location \( x \) in space. This element moves up and down in a simple harmonic motion. This motion, as we know, is the example of periodic motion, so let us see what the period of this motion is, and how it is related to the other parameters in the equation 1.1.1. Since the position of string’s element must be the same after the time interval equal to the period, we have
\begin{align*}
y(x,t) &= y(x,t+T), \\
y_m \sin(kx - \omega t) &= y_m \sin(kx - \omega(t+T)), \\
kx - \omega t - 2\pi &= kx - \omega(t+T) \\
2\pi &= \omega T.
\end{align*}

This means that
\[
\omega = \frac{2\pi}{T}.
\] (1.1.3)

The SI unit for the angular frequency is radian per second \((\text{rad/s})\). We can also define the linear frequency of wave as
\[
f = \frac{1}{T} = \frac{\omega}{2\pi}.
\] (1.1.4)

It is the number of oscillations made per unit of time and is usually measured in Hertz.

Waves move in space with certain velocity. Even though this motion is not the actual motion of matter from one point in space to another, but it has to do with the motion of the wave pattern. This wave pattern moves for distance \(\Delta x\) during the time interval \(\Delta t\), so we can introduce the wave’s speed \(\Delta x/\Delta t\) or \(dx/dt\). This wave’s speed is also known as the \textit{phase speed}, because it represents the motion of the point, where the wave's phase remains constant as it moves in the positive \(x\)-direction, which is
\[
kx - \omega t = \text{const}.
\]

Taking time derivative of this equation, we have
\[
k \frac{dx}{dt} - \omega = 0, \\
\frac{dx}{dt} = v = \frac{\omega}{k}.
\]

So the wave’s speed is
\[
v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f.
\] (1.1.5)

The last equation shows that this wave moves for a distance of one wavelength during the time interval equal to the wave's period.

So far we have considered waves moving in the positive direction of axis \(x\). We can also introduce waves moving in the negative \(x\)-direction, by replacing the sign in the equation 1.1.1, which is
\[
y = y_m \sin(kx + \omega t).
\] (1.1.6)
Exercise: Show that this wave is indeed moving in the negative $x$-direction (has the negative speed).

Let us now find the general equation valid for any type of waves. To do so, we will calculate the second derivative taken from the equations 1.1.1 and 1.1.6 with respect to time and space variables.

\[ \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial x^2} y_m \sin (kx \pm \omega t) = -y_m (\pm \omega)^2 \sin (kx \pm \omega t) = -\omega^2 y, \]

\[ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial x^2} y_m \sin (kx \pm \omega t) = -y_m k^2 \sin (kx \pm \omega t) = -k^2 y. \]

This means that

\[ \frac{\partial^2 y}{\partial t^2} = \omega^2 \frac{\partial^2 y}{\partial x^2}, \]

\[ \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}. \]

(1.1.7)

The last differential equation 1.1.7 is known as the wave equation. Any function, which represents solution of this equation, is a wave. The general solution of this equation is not necessarily harmonic function, but any function, which has a form

\[ y(x,t) = h(kx \pm \omega t), \]

where $h$ is an arbitrary function of the $kx \pm \omega t$ argument.

Now let us consider a special example of the wave traveling in the stretched string and see how the speed of this wave is related to the properties of the medium of this string. Since the elements of the string are oscillating, when the wave passes the medium, those elements should have both mass and elasticity to make these oscillations possible. So, the speed of this wave can be calculated in terms of the string’s elements mass and tension in the string.

At this point we can use a technique which we have learned during one of the first lectures last semester, known as dimensional analysis. Since we are looking for the speed, it should have the dimension of length divided by time

\[ v = \frac{l}{t}. \]

If we consider a small element of the string, we can introduce its mass by means of the linear mass density $\mu$, which has a dimension of mass per unit of length

\[ [\mu] = \left[ \frac{m}{l} \right]. \]
Finally to describe elasticity of the string, which takes place due to the stretching under the tension, we shall introduce tension \( \tau \), which has dimension of force

\[
[\tau] = \left[ \frac{ml}{t^2} \right].
\]

So we have

\[
v \propto \tau^p \mu^q,
\]

\[
\frac{l}{t} = \left( \frac{ml}{t^2} \right)^p \left( \frac{m}{l} \right)^q,
\]

1 = \( p - q \),

-1 = -2p,

0 = p + q,

and the solution of this set of equations is

\[
p = \frac{1}{2}, q = -\frac{1}{2},
\]

\[
v \propto \sqrt{\frac{\tau}{\mu}}.
\]

This means that the wave’s speed is proportional to the square root of tension divided by the mass density. But we cannot find the coefficient of proportionality just based on dimensional analysis, so we have to use Newton’s second law, to see what it is.

Consider the small element of the string, shown in the picture above. We choose a coordinate system which moves with constant speed of the wave in the same direction as the wave moves. So, we are always considering the top portion of the same pulse on the string. Let this element of the
string have length of $\Delta l$. There are two tension forces $\tau$ acting on both ends of this element shown in the picture. Each of these forces has vertical component directed to the center of the circle of radius $R$, shown in the picture. The net centripetal force acting on this element is

$$F = 2\tau \sin \theta \approx 2\tau \theta = \tau \frac{\Delta l}{R}.$$  

Here, we have used the fact that the angle and the element's length are small. Let $\Delta m$ to be the small mass of this element then, according to the Newton's second law

$$\Delta ma = F,$$

$$\Delta ma = \tau \frac{\Delta l}{R},$$

$$a = \frac{\tau}{R} \frac{\Delta l}{\Delta m} = \frac{\tau}{\mu} \frac{1}{R}.$$  

Since this acceleration is centripetal acceleration, we have

$$\frac{v^2}{R} = \frac{\tau}{\mu} \frac{1}{R},$$

$$v = \sqrt{\frac{\tau}{\mu}}. \quad (1.1.8)$$

So the unknown coefficient is equal to unity. The speed of the wave along the stretched ideal string depends only on its tension and the linear density of the string but not on the wave's frequency.

Waves are not associated with motion of the matter, but they are responsible for motion of energy through the medium. Let us find the energy associated with a wave. This is mechanical energy which can exist in two forms as kinetic energy and as potential energy. The kinetic energy is related to transverse motion of the string's elements. The potential energy is related to elastic properties of the string when string's element changes its length. As the pulse moves along the string, changing velocity and length of different elements of the string, it transfers energy along the string. Let us find the rate of the energy transmission. The string's element of mass $dm$ has kinetic energy

$$dK = \frac{1}{2} dm u^2,$$

where $u$ is the velocity associated with the transverse motion of this string’s element, which is

$$u = \frac{dy}{dt} = -y_m \omega \cos(\omega - \omega t).$$

If this element has length $dx$, then $dm = \mu dx$ and
\[ dK = \frac{1}{2} \mu dx y_m^2 \omega^2 \cos^2(kx - \omega t), \]
\[ \frac{dK}{dt} = \frac{1}{2} \mu \frac{dx}{dt} y_m^2 \omega^2 \cos^2(kx - \omega t), \]
\[ \frac{dK}{dt} = \frac{1}{2} \mu \nu y_m^2 \omega^2 \cos^2(kx - \omega t), \]

where \( v = \frac{dx}{dt} \) is the phase velocity. The average value of the \textit{cosine} squared function over integer number of periods is 1/2, so the average energy rate will be
\[ \frac{dK}{dt}_{\text{avg}} = \frac{1}{4} \mu \nu \omega^2 y_m^2. \]

The average rate of potential energy is the same, so the total power transmitted by the wave is twice as much, which is
\[ P_{\text{avg}} = \frac{1}{2} \mu \nu \omega^2 y_m^2. \] (1.1.9)

This power is proportional to the square of angular frequency and square of amplitude, which is the general result for any type of waves.

5. \textbf{Superposition of Waves}

Now we will consider a case, when there are several waves coming to the same point in space from different sources. This could be the case of the sound waves coming from different objects or the waves on the surface of water coming from different disturbances.

Since we were discussing transverse waves in the string, let us continue with this example and consider two waves traveling along the string simultaneously. In this case the net displacement of the string's element will be an algebraic sum of the two displacements due to each of the waves
\[ y'(x,t) = y_1(x,t) + y_2(x,t). \] (1.1.10)

This is known as the \textit{principle of superposition} of mechanical waves. The overlapping waves do not in any way alter each other.

The net wave depends on the extent to which the waves are in phase. If these two waves are exactly in phase, the net wave is just a doubled wave. However, the case, when there is a finite phase difference \( \phi \) between the waves, is different. We have
\[ y_1(x,t) = y_m \sin(kx - \omega t) \]
and
\[ y_2(x,t) = y_m \sin(kx - \omega t + \phi). \]
These two waves have the same frequency and the same wave number. They both travel in positive $x$ direction and have the same amplitude. They are only different by the phase constant $\phi$. In this case the net wave is going to be

$$y'(x,t) = y_1(x,t) + y_2(x,t) =$$

$$y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi) =$$

$$2y_m \cos\left(\frac{\phi}{2}\right) \sin\left(kx - \omega t + \frac{\phi}{2}\right) .$$

So the resultant wave is again a sinusoidal wave traveling in the same positive $x$ direction. However, this wave has different phase constant $\phi/2$ and different amplitude $y'_m = 2y_m \cos(\phi/2)$. In the case if the phase difference between the two original waves is zero the amplitude of the resultant wave achieves its maximum possible value $2y_m$, which is called fully constructive interference. If the original phase difference was equal $\pi$ then the amplitude of the resultant wave is zero and it is a case of fully destructive interference, like no wave goes along the string at all. In all other cases we have intermediate interference.

Now let us consider the case, when the two waves are traveling along the string in the opposite directions. Let these waves be of the same angular frequency and the same wave number. In that situation there are places on the string called nodes, where the string never moves. Half way between the adjacent nodes there are antinodes, where the amplitude of the resultant wave has its maximum. The pattern like this is called a standing wave, because this wave does not move in space, but just oscillate with time. Let us analyze this standing wave combined from

$$y_1(x,t) = y_m \sin(kx - \omega t)$$

and

$$y_2(x,t) = y_m \sin(kx + \omega t) ,$$

which gives

$$y'(x,t) = y_1(x,t) + y_2(x,t) =$$

$$y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) =$$

$$2y_m \sin(kx) \cos(\omega t) .$$

This equation does not describe a traveling wave. Instead it describes a standing wave. The absolute value of the first term in this equation $|2y_m \sin(kx)|$ gives the amplitude of oscillation at point $x$. As you can see this amplitude varies with position. At nodes it is equal zero.
\[
\sin(kx) = 0, \\
kx = \pi n, \ n = 0,1... \\
x = \frac{\pi n}{k} = n \frac{\lambda}{2}.
\]

The last equation shows positions of the nodes along axis \(x\). They are separated by distances equal to half of wavelength. The position of antinodes, where the amplitude has its maximum, can be found from the condition

\[
|\sin(kx)| = 1, \\
kx = \pi \left( n + \frac{1}{2} \right), \ n = 0,1... \\
x = \frac{\pi}{k} \left( n + \frac{1}{2} \right) = \left( n + \frac{1}{2} \right) \frac{\lambda}{2}.
\]

They are also separated by half of wavelengths.

The standing wave can be obtained as a result of the reflection of the traveling wave in the string from the boundary. Indeed, the reflected wave has the same frequency and wavelength, but travels in the opposite direction. If the end of the string is fixed (attached to the wall), then the standing wave will have a node at this point. If the end of the string is fasten to a light ring on a vertical rod, so it can move freely, then this point is going to be an anti-node for the resultant standing wave.

In the case if both ends of the string are fixed (the guitar string for instance) we can obtain a series of the standing waves at certain frequencies. Such standing waves are said to be produced at resonance and the string is resonating at these resonate frequencies. Since both ends of the string are fixed, they produce nodes for the resultant standing wave. This is only possible if the distance \(L\) between the ends of the string is equal the integer number of the half wavelengths, which is

\[
\lambda = \frac{2L}{n}, \ n = 1,2...
\]

So this wave has frequencies

\[
f = \frac{v}{\lambda} = n \frac{v}{2L}, \ n = 1,2...
\]

The oscillation mode with the lowest frequency \(f=\frac{v}{2L}\) is called the fundamental mode or the first harmonic. The collection of all possible modes for \(n=1,2...\) is called the harmonic series and \(n\) is called the harmonic number.