Lecture 1.2

Sound

The most well known example of mechanical wave is a sound wave. A sound wave in the air is a longitudinal spherical wave. Sound waves can exist in many types of media: liquids, solids and gases. Sound waves in water can be used by submarines to detect positions of the unknown objects. Sound waves in solid substances, such as ground, are used to study seismic activity and other properties of the Earth's crust. The simplest type of these waves is a sound wave in air.

In kinematics we used an idea of the point or particle-like object. The size of this object is much smaller than the distance for which it travels. Using the same idea, we can introduce a point-like source of sound. For instance, a speaker can be treated as a point-like source of sound compared to the size of the auditorium. In this case sound waves are propagating in all directions from the speaker, so everybody can hear. This is different compared to the situation of the string where transverse wave travels in one direction $x$ only. In the case of the string a wavefront, a surface where oscillations of matter have the same phase, is a planar surface. In the case of the sound wave, traveling in all directions along the rays in the air, the wave fronts (perpendicular to the rays) are spherical. This is a longitudinal wave, so that oscillations of air are also taking place in the direction of rays (perpendicular to the wavefronts) while for transverse wave they took palace in the direction of the wavefront (perpendicular to the rays). On the other hand, if we consider a spherical wave at a very large distance from its source, the wave front’s curvature is almost negligible, and one can treat the wave as planar.

We saw last time that the speed of the transverse wave in the medium depends on both the inertial properties of the medium and its elastic properties. In the case of the string, the inertial property associated with its kinetic energy (it was the linear mass density $\mu$) and elastic property associated with its potential energy (it was tension in the string $\tau$). This general idea remains valid for any type of wave, but the particular choice of variables varies.

Let us see how one can find the speed of sound in air. Since this wave is propagating in three dimensions in space, we have to replace the linear density by the volume density, $\rho$ (mass of the air per unit of volume). The elastic properties of air are
related to periodic compressions and expansions of the small volume elements of air. The bulk modulus
\[ B = -\frac{\Delta p}{\Delta V/V} \]  \hspace{1cm} (1.2.1)
is responsible for these changes. In this equation \( \Delta p \) is the change of pressure and \( \Delta V/V \) is the fractional change of volume. \( B \) is always positive quantity, because for any stable substance the increase of pressure causes decrease of volume. \( B \) has the same dimension as pressure, which is dimension of force per unit of area. Following dimensional analysis one can obtain the equation for the speed of sound
\[ v = \sqrt{\frac{B}{\rho}}, \]  \hspace{1cm} (1.2.2)
which is actually correct even with the correct coefficient of proportionality.

**Exercise:** Prove equation 1.2.2 based on dimensional analysis.

Even though equation 1.2.2 is the right equation, we still have to prove it, not only based on dimensional analysis, which does not provide the coefficient of proportionality, but based on the Newton's laws. To make this derivation simpler we shall only consider a sound wave traveling in the straight tube filled by air, so it becomes a planar sound wave. We shall use the same method which we have used last time, when obtaining the speed of the transverse wave. This means we will consider one pulse going along the tube and we choose the reference frame, which travels along with this pulse at the same speed \( v \) in the positive \( x \) direction along the tube. If the pressure of undisturbed air in the tube was \( p \), then the pressure inside of the pulse is \( p + \Delta p \). In our reference frame the air inside of the tube travels towards the pulse with speed \( v \). We shall consider the slice of air with cross section aria \( A \) and thickness \( \Delta x \). When this slice of air enters the pulse, it changes its velocity from \( v \) to \( v + \Delta v \), where \( \Delta v \) is negative since air slows down under the influence of the positive pressure \( \Delta p \). This slowing will take place during the time while the entire air slice enters the pulse. It takes \( \Delta t = \Delta x/v \). This change of speed occurs under the influence of force. The net force acting on the air slice is the difference between the force acting on the front face and the force acting on the rear face, which is \( F = pA - (p + \Delta p)A = pA - pA - \Delta pA = -\Delta pA \). The “minus” sign of the net force means that it is acting in the direction opposite to the velocity of the air element, causing it to
slow down. The mass of this slice of air is \( \Delta m = \rho A \Delta x \) and its acceleration is \( a = \frac{\Delta v}{\Delta t} \), so the Newton's second law for this air element gives

\[
F = \Delta ma,
\]

\[-\Delta p A = \rho A \Delta x \frac{\Delta v}{\Delta t},
\]

\[-\Delta p = \rho v \Delta t \frac{\Delta v}{\Delta t},
\]

\[-\Delta p = \rho v \Delta v,
\]

\[\rho v^2 = -\frac{\Delta p}{\Delta v/v}.\]

The air, which had volume \( V = A \Delta x = A v \Delta t \) outside of the pulse, is compressed by \( \Delta V = A \Delta v \Delta t \), when it enters the pulse, so

\[
\frac{\Delta V}{V} = \frac{A \Delta v \Delta t}{A v \Delta t} = \frac{\Delta v}{v}
\]

and

\[
v^2 = \frac{\frac{\Delta p}{\Delta v/v}}{\rho} = -\frac{\Delta p}{\rho \frac{\Delta V/V}{\rho}} = \frac{B}{\rho},
\]

\[v = \sqrt{\frac{B}{\rho}},\]

which proves the equation 1.2.2.

Let the sound wave in the tube filled with air appear due to the motion of the piston at the end of the tube which moves from the left to the right and backwards in the sinusoidal pattern. This motion will cause the density change of the air elements next to the piston according to the same sinusoidal law. Then these changes of density will travel along the tube as sound. Each element of air with thickness \( \Delta x \) will oscillate according to the same sinusoidal law as the piston does. In longitudinal wave, the oscillations take place in the same direction \( x \) as this wave propagates. To avoid confusion we will use letter \( s \) to show this displacement. Let this wave have a form of the cosine-like function,

\[
s(x,t) = s_m \cos(kx - \omega t),
\]

where \( s_m \) is the displacement’s amplitude, which is the maximum displacement of the air element from equilibrium position. All other characteristics of this wave, such as \( k, \omega, f, \lambda, T \) are defined in a same way as they were in the case of the transverse wave.
Let us find how air’s pressure will change along the wave. Again we shall consider the air element of length $\Delta x$, cross section area $A$ and volume $V = A\Delta x$. When this element is displaced, it changes its volume by $\Delta V = A\Delta s$, where $\Delta s$ is the change of the displacement of two faces of the element. Thus, using the definition of the bulk modulus 1.2.1, we have

$$
\Delta p = -B \frac{\Delta V}{V} = -B \frac{A\Delta s}{A\Delta x} = -B \frac{\Delta s}{\Delta x} = -B \frac{\partial s}{\partial x},
$$

$$
\Delta p = -B \frac{\partial s}{\partial x} (s_m \cos(kx - \omega t)) = B s_m k \sin(kx - \omega t). \quad (1.2.4)
$$

This means that the change of pressure along the wave takes place in the opposite phase compared to the change of displacement. The amplitude of pressure change is

$$
\Delta p_m = B k s_m = v^2 \rho k s_m = v\rho \omega s_m. \quad (1.2.5)
$$

Like any other waves sound waves can undergo interference. This, for instance, takes place, when you hear sounds coming from different speakers. For simplicity we shall consider the interference between the two identical sound waves traveling in the same direction.

This situation is shown in the picture, where two waves are traveling from the two point-like sources $S_1$ and $S_2$ to the same point in space $P$. Both waves are in phase and of identical wave length. We can also assume that distances from the sources to point $P$ are much larger than the distance between the sources themselves, so we can say that waves are traveling in the same direction. Since distance $L_1$ traveled from the first source is slightly different from the distance $L_2$ traveled from the second source, the answer for the question, whether or not these waves are in phase at point $P$, depends strongly on the path length difference $\Delta L = |L_2 - L_1|$. The phase difference between the two waves is equal

$$
\phi = \frac{\Delta L}{\lambda} 2\pi. \quad (1.2.6)
$$
The fully constructive interference occurs if the phase difference is equal to the integer number of $2\pi$, so
\[
\frac{\Delta L}{\lambda} = 0, 1, 2, \ldots
\]

Fully destructive interference occurs when the phase difference between the waves is equal the odd multiple of $\pi$, so
\[
\frac{\Delta L}{\lambda} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots
\]

Another important characteristic of sound wave is its intensity, which actually shows how strong the sound is at a given point. For instance, if you cannot sleep at night, while somebody plays a loud music, this is because the intensity of this sound is too high. The intensity, $I$, of a wave at certain surface is the average power per unit area transferred by this wave through or onto the surface
\[
I = \frac{P}{A}. \quad (1.2.7)
\]

If we consider a point-like source of sound, which emits sound waves isotropically (with equal intensity in all directions) and if the sound wave's energy is conserved (it is not dissipated in space) then we can write that
\[
I = \frac{P_s}{4\pi r^2}. \quad (1.2.8)
\]

Here $P_s$ is the power of the source and $r$ is the distance from the source. As you can see, the intensity of the sound gets smaller with distance, not even because of reflections and scattering from different objects, but because the same energy spreads over larger and larger areas of spheres $4\pi r^2$ surrounding this point-like source.

When talking about sound we usually mean the sound range, which can be heard by an ordinary human ear. However, this rage is enormous. Human ear can distinguish between the sounds, with intensities varying up to $10^{12}$ times. This is why it is not very convenient to use intensities on such a huge scale. Instead people use a logarithmic scale to deal with this large range. Let us introduce a new quantity, known as the intensity level or sound level, which is
\[
\beta = (10dB)\log\frac{I}{I_0}. \quad (1.2.9)
\]
Here $dB$ is the abbreviation for *decibel*, the unit of sound level named after Alexander Graham Bell. $I_0 = 10^{-12} \text{W/m}^2$ stands for the standard reference intensity, chosen because it is near the law limit of the human range of hearing. This standard reference level corresponds to zero decibels. If sound intensity increases 10 times, the sound level only increases by 10 decibels.

Let us see how intensity of sound is related to other characteristics of sound waves, such as its amplitude, velocity and frequency. We will follow the same method which we have already considered many times before, we can choose a slice of air of thickness $dx$, cross section area $A$ and mass $dm$, which oscillates back and forth in the direction of the wave’s propagation. The kinetic energy of this slice is

$$dK = \frac{1}{2} dm v_s^2,$$

where $v_s$ is the speed of oscillating element of air (not the speed of the wave), which is

$$v_s = \frac{\partial s}{\partial t} = -\varphi m \sin (kx - \omega t).$$

Taking into account that $dm = \rho A dx$, we have

$$dK = \frac{1}{2} \rho A dx (-\varphi m)^2 \sin^2 (kx - \omega t).$$

So the kinetic energy of this element will change at the rate

$$\frac{dK}{dt} = \frac{1}{2} \rho A v_o^2 s_m^2 \sin^2 (kx - \omega t),$$

where we used $v = \frac{dx}{dt}$, the speed of this wave. The average rate at which kinetic energy is transported by the wave can be calculated by averaging the last equation over time. Taking into account that average value of *sine* squared function over integer number of periods is $1/2$, we have

$$\left( \frac{dK}{dt} \right)_{avg} = \frac{1}{4} \rho A v_o^2 s_m^2.$$

Potential energy is carried along the wave at the same average rate, so the wave intensity is going to be

$$I = \frac{(dK/dt)_{avg} + (dU/dt)_{avg}}{A} = \frac{2(dK/dt)_{avg}}{A} = \frac{1}{2} \rho v_o^2 s_m^2.$$

(1.2.10)
One of the most interesting types of sound is the musical sound. In the most part of cases musical sound is obtained as a result of oscillations of some part of a musical instrument. For instance, it could be oscillating string of the string instrument such as a guitar or a violin. Actually, in this case it is even more complicated, because not only strings but also the bodies of these musical instruments are oscillating. In the case of the percussion like drums or plates, the oscillating parts are different membranes of the instruments. In the case of winds, the oscillating parts are not instruments themselves, but the air columns inside of those instruments.

We have already discussed oscillations of strings. They can occur as a result of the standing transverse wave in those strings. The wavelength required for such a standing wave is the one, which has frequency corresponding to the resonant frequency of the string. In the case of the standing wave, the string will oscillate with a large enough amplitude to produce a strong hearable sound wave in the air.

In a same way as we can set up the standing wave in a string, we can do it in a pipe of some wind instrument (for instance organ pipe) filled with air. In this case it is a longitudinal sound wave, but again it can be obtained as a result of the reflection from the end of the pipe. This reflection will take place for the pipe with the open end as well as for the pipe with the closed end. In order to have this standing wave, the wavelength should be in certain relation to the length of the pipe. These wavelengths correspond to the resonant frequencies of the pipe. The standing wave in the pipe is very similar to the standing wave in a string. The nodes of the standing wave correspond to the closed end of the pipe and antinodes correspond to the open end of the pipe.

In the case of the pipe with two open ends, there are two antinodes at those ends. The simplest standing wave, which one can set up in such a pipe, is a wave, which only has one node in between the ends. This is fundamental mode or the first harmonic of that pipe. It is easy to see that its wavelength \( \lambda = 2L \), where \( L \) is the length of the pipe. More generally for the pipe with two open ends the resonant frequencies occur, when

\[
\lambda = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots
\]

where \( n \) is called harmonic number. The resonant frequencies in this case are

\[
f = \frac{v}{\lambda} = \frac{nv}{2L}, \quad n = 1, 2, 3, \ldots
\]
If one has a pipe with only one end opened (which will be the case in your laboratory experiment next time), it will be slightly different, since it should be a node at one end and antinode at the other. In the simplest case a wavelength should be $4L$ which is fundamental mode for this pipe. In general

\[ \lambda = \frac{4L}{n}, \quad n = 1,3,5... \]

In this case harmonic number $n$ is an odd number and frequencies are given by

\[ f = \frac{v}{\lambda} = \frac{nv}{4L}, \quad n = 1,3,5... \]

So, the length of musical instrument reflects the possible values of frequencies it can generate. The larger the length, the lower frequency that it can play.

If one listens to sounds coming from two different sources, but those sources have very close frequencies, it will be almost impossible to distinguish between the two sounds. Instead the person will hear a sound which has a frequency equal to the average of the two combined frequencies. Intensity of this combined sound will change slightly with time wavering *beats* with frequency equal to the difference between the two combined frequencies. Let us consider this beat phenomenon at some point in space, where waves coming from the two sources have the form of

\[ s_1 = s_m \cos \omega_1 t, \]
\[ s_2 = s_m \cos \omega_2 t, \]

where $\omega_1 > \omega_2$. The superposition principle of two waves gives

\[ s = s_1 + s_2 = s_m \left( \cos \omega_1 t + \cos \omega_2 t \right) = 2s_m \cos \left( \frac{\omega_1 - \omega_2}{2} t \right) \cos \left( \frac{\omega_1 + \omega_2}{2} t \right), \]

so we have

\[ s(t) = 2s_m \cos \omega' t \cos \omega t, \]

where $\omega' = \frac{\omega_1 - \omega_2}{2}$ and $\omega = \frac{\omega_1 + \omega_2}{2}$. Since frequencies of the two waves are very close, this means that $\omega \gg \omega'$ and we can treat the first cosine function in the last equation as the slowly changing wave's amplitude. The angular frequency at which beats occur is $\omega_{\text{beat}} = 2\omega' = \omega_1 - \omega_2$. This beat-phenomenon is used by musicians to turn their instruments on the standard frequency until beats disappear.
The last question which we shall address today is how sound changes if the source of sound and/or the observer (who receives this sound) are in the state of the relative motion. You can observe this effect in everyday life. If some emergency vehicle with its siren on passes you on the street, you may notice that the sound of this siren is very different before and after the car passes you. It changes frequencies depending on whether or not the car is moving towards you or away from you. This effect is known as the Doppler effect since it was first studied by the Austrian physicist Johann Christian Doppler in 1842. Even though this effect works for any type of waves, here we shall only consider it for the sound waves in air. The air itself will be considered as a reference frame. So, we will measure velocity of the source of sound, $S$, as well as of the detector, $D$, relative to the air in which sound is traveling. For simplicity, we will assume that $S$ and $D$ are moving along the straight line with speed less than the speed of sound in the air. In this case the emitted frequency of sound $f$ is related to the detected frequency $f'$ as

$$f' = f \frac{v \pm v_D}{v \pm v_S},$$

(1.2.11)

where $v$ is speed of sound, $v_D$ is the detector's speed relative to the air and $v_S$ is the speed of the sound source relative to the air. When the motion of the detector or the source is towards each other, the sign of the speed must give an upward shift in frequency. When the motion of the detector or source is away from each other, the sign of speed must give the downward shift in frequency. In the example, which I have mentioned, when the vehicle is moving towards you, you hear the higher frequency compared to that which was actually emitted. When it is moving away from you, you hear the lower frequency.

Let us consider two special cases

1) The detector moves relative to the air and the source is stationary.

Let this detector be moving towards the stationary source, which emits a spherical sound wave (in all directions) with speed $v$, wavelength $\lambda$ and frequency $f$. If the source were stationary, the wave fronts would move for time $t$ for distance $vt$. In this case the number of wave fronts detected by the detector for this time is $vt/\lambda$, so the detected frequency is

$$f = \frac{vt}{\lambda t} = \frac{v}{\lambda},$$

as it should be for the real wave. If, however, the detector is moving towards the source, their relative velocity $v + v_D$ and so the detected frequency is going to be

$$f' = f \frac{v \pm v_D}{v \pm (v + v_D)}.$$
\[ f' = \left( \frac{v + v_D}{\lambda_f} \right) \cdot \frac{v + v_D}{v} = f' = \frac{v + v_D}{v} \]. If detector moves away from the source, then the relative velocity is going to be \( v - v_D \) and we have \( f' = f \frac{v - v_D}{v} \). So the equation 1.2.11 is proven for the moving detector and stationary source.

2) Source moves relative to the air and the detector is stationary.
Let the source move towards D with speed \( v_S \) and emit wave with frequency \( f \). So it takes time \( T = \frac{1}{f} \) between the emissions of the two wave fronts. For this time S moves for distance \( v_ST \) and the wave front emitted before for distance \( vT \). This means that a wavelength is detected by D (which is distance between two wave fronts) will be

\[ \lambda' = vT - v_ST, \quad f' = \frac{v}{\lambda' vT - v_ST} = \frac{v}{v/f - v_S/f} = f \frac{v}{v-v_S}. \]

If the source moves in opposite direction we need to change the sign of its velocity. So, the equation 1.2.11 is also proven for the moving source.

The general Doppler equation is just combination of these two cases. However, it only works for the speeds of sources smaller than the speed of sound. Indeed the detected frequency becomes infinite, if the source moves with the speed of sound, which means it keeps pace with its own spherical wave front. In the case of the supersonic speed the source will move faster than its own sound wave and it will create a cone of wave fronts known as the Mach cone. A shock wave exists along the surface of this cone, producing a burst of sound known as a sonic boom.