The potential wells

Physics 4330, Lecture 12

The time-independent Schrodinger Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u(x)}{\partial x^2}+V(x)u(x)=Eu(x),$$
$$Hu(x)=Eu(x),$$
$$\frac{\partial^2 u(x)}{\partial x^2}+\frac{2m}{\hbar^2}\big(E-V(x)\big)u(x)=0$$

The potential barrier

$$V(x)=0,\ x<-a$$
$$=V_0,\ -a<x<a,$$
$$=0,\ x>a$$

Solution of the Schrodinger Equation

$$x < -a,$$
$$u(x) = A\left(e^{ikx} + Re^{-ikx}\right),$$
$$-a < x < a,$$
$$u(x) = A\left(Be^{-\kappa x} + Ce^{\kappa x}\right),$$
$$a < x,$$
$$u(x) = ATe^{ikx}$$

Relations between coefficients

$$R = ie^{-2ika}\frac{-\left(\kappa^2 + k^2\right)\sinh 2\kappa a}{2k\kappa\cosh 2\kappa a - i\left(-\kappa^2 + k^2\right)\sinh 2\kappa a},$$
$$T = e^{-2ika}\frac{2k\kappa}{2k\kappa\cosh 2\kappa a - i\left(-\kappa^2 + k^2\right)\sinh 2\kappa a},$$

$$|T|^2 = \frac{(2k\kappa)^2}{(2k\kappa)^2 + \left(\kappa^2 + k^2\right)^2\sinh^2 2\kappa a},$$

Special cases

$$\kappa a >> 1,$$

$$|T|^2 \to \left(\frac{4k\kappa}{k^2 + \kappa^2} \right)^2 e^{-4\kappa a}$$

The WKB Approximation

$$E > V(x),$$

$$u(x) = R(x) \exp\left(i \int_{x_1}^{x} dy \sqrt{2m\left(E - V(y)\right)/\hbar^2} \right),$$

$$E < V(x),$$

$$u(x) = R(x) \exp\left(-\int_{x_1}^{x} dy \sqrt{2m\left(V(y) - E\right)/\hbar^2} \right),$$

$$|T|^2 = C \exp\left(-2 \int_{barrier} dy \sqrt{2m\left(V(y) - E\right)/\hbar^2} \right)$$

Examples

- Cold emission

$$|T|^2 = C e^{-\frac{4}{3} a \sqrt{\frac{mW}{\hbar^2}}}$$

- Nuclear physics

Bound States in a Potential Well

$$V(x) = 0, \; x < -a$$
$$= -V_0, \; -a < x < a,$$
$$= 0, \; a < x$$

Solution of the Schrodinger Equation

$$x < -a,$$
$$u(x) = C_1 e^{\alpha x},$$
$$-a < x < a,$$
$$u(x) = A\cos qx + B\sin qx,$$
$$a < x,$$
$$u(x) = C_2 e^{-\alpha x}$$

Even and Odd solutions

- Boundary conditions
- Even $\alpha = q\tan qa$
- Odd $\alpha = -q\cot qa$

Even and Odd solutions

$$\lambda = \frac{2mV_0a^2}{\hbar^2},$$

$$y = aq = a\sqrt{\frac{2m}{\hbar^2}\left(V_0 - |E|\right)},$$

$$\text{even} \qquad \tan y = \frac{\sqrt{\lambda - y^2}}{y}, \quad y \approx \left(n + \frac{1}{2}\right)\pi, \; n = 0,1,2...$$

$$\text{odd} \qquad -\cot y = \frac{\sqrt{\lambda - y^2}}{y}, \quad y \approx n\pi, \; n = 1,2,3...$$

Double Well

- If wells are very far apart

$$u_{even}(x) = \frac{1}{N_e}\left(u_R(x) + u_L(x)\right)$$

$$u_{odd}(x) = \frac{1}{N_o}\left(u_R(x) - u_L(x)\right)$$

Delta Function Potential

$$V(x) = -\frac{\hbar^2\lambda}{2ma}\delta(x)$$

Solution of the Schrodinger Equation

$$x<0,$$
$$u(x)=Ce^{\kappa x},$$
$$x>0,$$
$$u(x)=Ce^{-\kappa x}$$

Double Delta Function Potential

$$V(x)=-\frac{\hbar^2\lambda}{2ma}\left[\delta(x-a)+\delta(x+a)\right]$$

Solution of the Schrodinger Equation

even

$$u(x)=Ce^{-\kappa x}, x>a$$
$$u(x)=CA\cosh\kappa x, -a<x<a$$
$$u(x)=Ce^{\kappa x}, x<-a,$$

odd

$$u(x)=Ce^{-\kappa x}, x>a$$
$$u(x)=CA\sinh\kappa x, -a<x<a$$
$$u(x)=-Ce^{\kappa x}, x<-a,$$