The Operator method in Quantum Mechanics

Physics 4330, Lecture 17

Arbitrary Observable

\[ Au_a(x) = au_a(x), \]
\[ \int_{-\infty}^{+\infty} du_a^*(x)u_a(x) = \delta_{a,a'}, \]
\[ \psi(x) = \sum_a C_a u_a(x), \]
\[ C_a = \int_{-\infty}^{+\infty} du_a^*(x)\psi(x), \]
\[ \sum_a |C_a|^2 = 1 \]

Properties of Hermitian operators

\[ A = A^*, \]
\[ \int_{-\infty}^{\infty} d\phi^*(x) A\psi(x) = \int_{-\infty}^{\infty} dx (A\phi(x))^* \psi(x) \]
Dirac notation

- **Ket notation**
  \[ \psi \]
  \[ a \]
  \[ b \]
  \[ a, b \]

Dirac notation

- **Bra notation**
  \[ \langle \psi \rangle \]
  \[ \langle a \rangle \]
  \[ \langle b \rangle \]
  \[ \langle a, b \rangle \]

Scalar Product

\[ \langle \phi | \psi \rangle, \]
\[ \langle \phi | \psi \rangle^* = \langle \psi | \phi \rangle, \]
\[ \langle \phi | \alpha \psi_1 + \beta \psi_2 \rangle = \alpha \langle \phi | \psi_1 \rangle + \beta \langle \phi | \psi_2 \rangle \]
Operators

\[ A |\psi\rangle = |A\psi\rangle \]

Hermitian Conjugate

\[ \langle A\phi|\psi \rangle = \langle \phi |A^+|\psi \rangle, \]
\[ \langle \psi |A\phi \rangle = (A\phi|\psi \rangle)^* = \langle \phi |A^+|\psi \rangle^* \]

Completeness

\[ |\psi\rangle = \sum_n C_n |n\rangle, \]
\[ \langle m|n \rangle = \delta_{mn}, \]
\[ C_n = \langle n|\psi \rangle, \]
\[ |\psi\rangle = \sum_n |n\rangle \langle n|\psi \rangle, \]
\[ \sum_n |n\rangle \langle n| = I \]
Operator of Position

Different representations of wave function

\[ \psi(x) = \langle x | \psi \rangle, \]
\[ \phi(p) = \langle p | \psi \rangle \]

Projection Operators

\[ P_n = |n \rangle \langle n |, \]
\[ P_n P_m = |m \rangle \langle m | n \rangle \langle n | = \delta_{mn} |n \rangle \langle n | = \delta_{mn} P_n, \]
\[ \sum_n P_n = I, \]
\[ P_n^2 = P_n. \]
Harmonic Oscillator

The Schrödinger Equation

\[ H |E\rangle = E |E\rangle, \]

\[ H = \frac{p_{op}^2}{2m} + V(x_{op}), \]

\[ \langle x | H |E\rangle = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \langle x | E \rangle + V(x) \langle x | E \rangle = E \langle x | E \rangle \]

The time-dependent Schrödinger Equation

\[ i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \]
Solution of Time-Dependent Schrödinger Equation

\[ |\psi(t)\rangle = \exp(-iHt/\hbar)|\psi(0)\rangle, \]
\[ \exp(-iHt/\hbar) = \sum_{n} \frac{1}{n!}(-iHt/\hbar)^n \]

Time Evolution of Physical Quantities

\[ \langle B \rangle_t = \langle \exp(-iHt/\hbar)|\psi(0)\rangle B \exp(-iHt/\hbar)|\psi(0)\rangle = \]
\[ = \langle \psi(0)\exp(iHt/\hbar)B \exp(-iHt/\hbar)|\psi(0)\rangle = \]
\[ = \langle \psi(0)|B(t)|\psi(0)\rangle \]

Time Evolution of Physical Quantities

\[ B(t) = \exp(iHt/\hbar)B \exp(-iHt/\hbar), \]
\[ \frac{d}{dt}B(t) = \frac{i}{\hbar}H \exp(iHt/\hbar)B \exp(-iHt/\hbar) \]
\[ - \frac{i}{\hbar} \exp(iHt/\hbar)B \exp(-iHt/\hbar)H = \]
\[ = \frac{i}{\hbar}(HB(t) - B(t)H) = \frac{i}{\hbar}[H, B(t)] \]
Examples