Homework 1

1. A bead slides on a smooth rod, the rod being constrained to rotate uniformly at an angular velocity $\omega$ in horizontal plane about an axis passing through one end perpendicular to the rod. Find the position of the bead on the rod as function of time, and the reaction of the rod on the bead. Solve the problem using a non-inertial reference frame attached to the rod.

2. If a projectile is fired due east from a point on the earth’s surface at a northern latitude $\lambda$, with initial speed $v_0$ and at the angle of inclination to the horizontal of $\alpha$. Show that the lateral deflection when the projectile strikes the earth is

$$d = \frac{4v_0^3}{g} \omega \sin \lambda \sin^2 \alpha \cos \alpha,$$

where $\omega$ is the angular speed of the earth’s rotation. If the range of the projectile for $\omega = 0$ were $R$ then show that the change of this range due to the rotation of the earth is

$$\Delta R = \sqrt{\frac{2R^3}{g} \omega \cos \lambda \left( \frac{1}{\tan \frac{\lambda}{2}} - \frac{1}{3} \tan \frac{\lambda}{2} \alpha \right)}.$$

3. One way of overcoming health hazards in the prolonged absence of a gravitational field on board of a spaceship is to produce an artificial gravity by spinning the ship about its axis. If, for example, a ship were built in the shape of a large ring, its outer wall would become a “floor” as the ring spun about its axis. The artificial gravitational field in this case would result from centrifugal acceleration. a) Estimate the angular velocity $\omega$ of spin required to produce an artificial gravitational field equivalent to that on the earth for a ring-shaped spaceship with the outer radius of $r_0 = 2000$ ft. b) The origin of the non-inertial reference frame is permanently fixed to the ship’s floor with the axis $x_3$ pointing “up” (towards the center of the ring), $x_2$-axis being tangential to the ring and $x_1$-axis being parallel to the axis of the ring. Find acceleration of the origin of this coordinate system. c) If $\vec{r}$ is the displacement vector of a particle extended from the origin of this non-inertial reference frame, what is the Lagrangian that gives equations of motion of a particle with respect to this accelerated coordinate system. Show that this Lagrangian can be reduced to

$$L = \frac{1}{2} m \left( \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) + \frac{1}{2} m \omega^2 \left( x_1^2 + x_2^2 \right) - m r_0 x_3 \omega^2 + m \omega \left( x_3 \dot{x}_2 - \dot{x}_2 x_3 \right).$$

d) Assuming that there are no constraints, so that the particle has three degrees of freedom, show that Lagrange’s equations of motion for that particle are

$$\ddot{x}_1 = 0,$$

$$x_2 - \omega^2 x_2 - 2\omega \dot{x}_1 = 0,$$

$$\ddot{x}_3 + \left( r_0 - x_3 \right) \omega^2 + 2\omega \dot{x}_2 = 0.$$
Use the method of complex variables to combine the last two equations into one equation for variable \( z = x_2 + ix_3 \) and find the general complex solution of that equation. e) Suppose that a particle on this spaceship is dropped from rest at height \( h \) above the floor. For these initial conditions determine the constants in the general solution of the equation for \( z \), you found in part d. f) Suppose that motion is over a small distance, so that \( \omega t \ll 1 \). Show that by expending your solution into Taylor series one can find

\[
x_2 = -\frac{1}{3} g \omega t^3 + \frac{1}{30} g \omega^3 t^5 + ..., \\
x_3 = h - \frac{1}{2} g t^2 + \frac{1}{8} g \omega^2 t^4 + ..., 
\]

where \( g = (r_0 - h) \omega^2 \) is artificial acceleration due to gravity. If \( \omega = 0.4s^{-1}, g = 32 \text{ ft/s}^2, h = 8 \text{ ft} \), solve for time it will take for this particle to hit the floor. Then determine \( x_2 \) (horizontal deflection of the particle). Notice that this deflection is quite large compared to the case of a body falling near the surface of the earth.